软件分析

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Static Program Analysis

Data Flow Analysis — Foundations

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Contents (II)

Let us first recall the iterative algorithm for data flow analysis

This general iterative algorithm produces a solution to data flow analysis

Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block B)

OUTPUT: IN[B] and OUT[B] for each basic block B

METHOD:

OUT[*entry*] = ∅;
for (each basic block $B \backslash entry$)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block $B \backslash entry$) {
        IN[B] = $\bigcup_{P \text{ a predecessor of } B}$ OUT[P];
        OUT[B] = $gen_B \cup (\text{IN}[B] - kill_B)$;
    }

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- Assume the domain of the values in data flow analysis is V, then we can define a k-tuple

$$(\mathrm{OUT}[n_1], \mathrm{OUT}[n_2], \ldots, \mathrm{OUT}[n_k])$$

as an element of set $(V_1 \times V_2 \ldots \times V_k)$ denoted as V^k , to hold the values of the analysis after each iteration.

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- Each iteration can be considered as taking an action to map an element of V^k to a new element of V^k, through applying the transfer functions and control-flow handing, abstracted as a function $F: V^k \to V^k$
- Then the algorithm outputs a series of k-tuples iteratively until a k-tuple is the same as the last one in two consecutive iterations

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Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

init → $(\perp, \perp, \ldots, \perp)$

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$$\textit{iter } 2 \rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$$

$$\vdots$$

$$\textit{iter } i \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$$

$$\textit{iter } i+1 \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$$

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Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

$$\begin{array}{lcl}
\textit{init} & \rightarrow & (\bot,\ \bot,\ \ldots,\ \bot) = X_0 \\
\textit{iter 1} & \rightarrow & (v_1^1, v_2^1, \ldots, v_k^1) = X_1 \\
\textit{iter 2} & \rightarrow & (v_1^2, v_2^2, \ldots, v_k^2) = X_2 \\
 & & \vdots \\
\textit{iter i} & \rightarrow & (v_1^i, v_2^i, \ldots, v_k^i) = X_i \\
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$F: V^k \rightarrow V^k$

$$
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\textit{iter 2} & \rightarrow (v_1^2, v_2^2, \ldots, v_k^2) & = X_2 = F(X_1) \\
 & \vdots & \\
\textit{iter i} & \rightarrow (v_1^i, v_2^i, \ldots, v_k^i) & = X_i = F(X_{i-1}) \\
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To answer these questions, let us learn some math first

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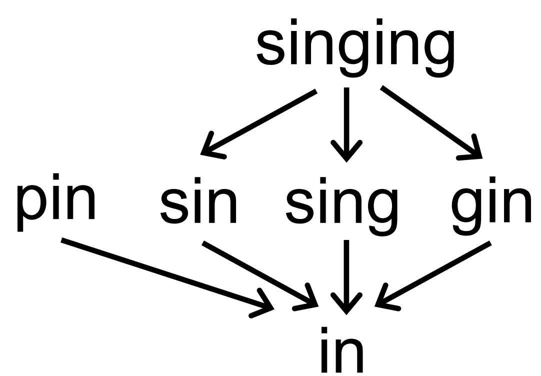
Example 3. Is $(S, \sqsubseteq)$ a poset where S is a set of English words and $\sqsubseteq$ represents the *substring* relation, i.e., s1 $\sqsubseteq$ s2 means s1 is a substring of s2?



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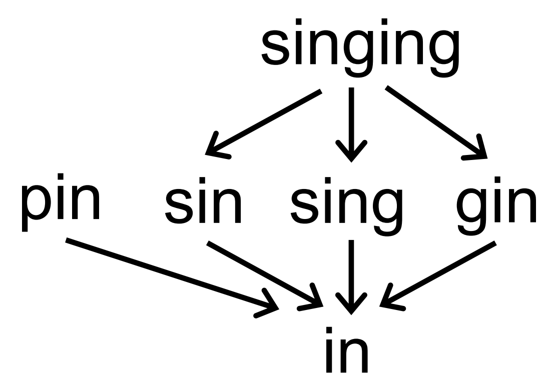

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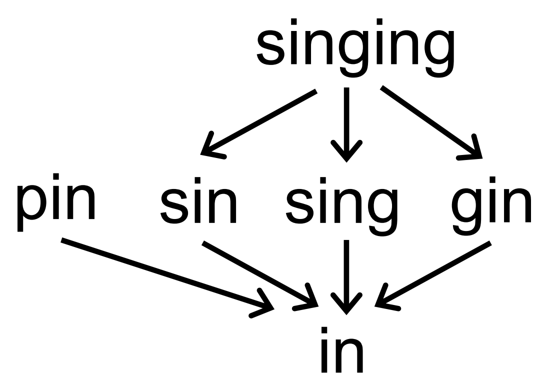

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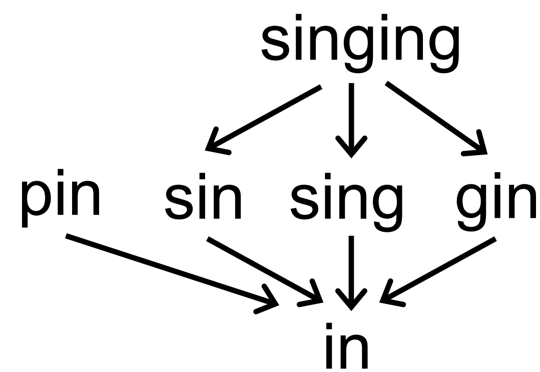

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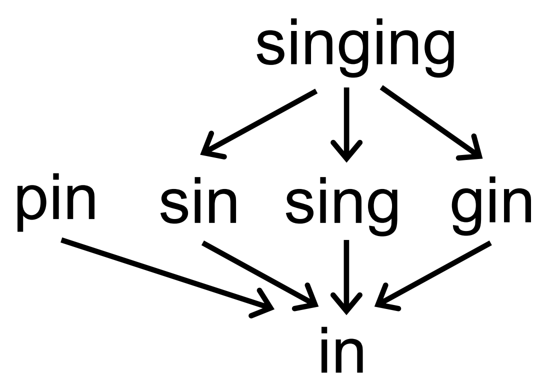

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partial means for a pair of set elements in P, they could be incomparable; in other words, not necessary that every pair of set elements must satisfy the ordering $\sqsubseteq$

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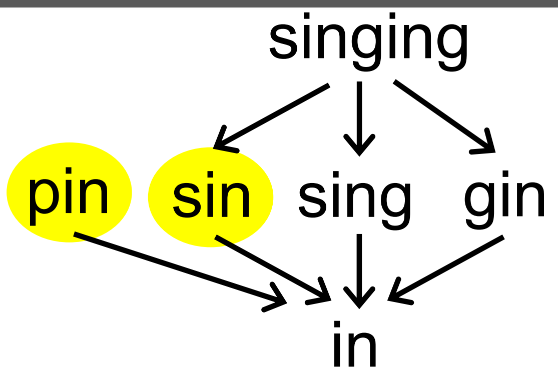

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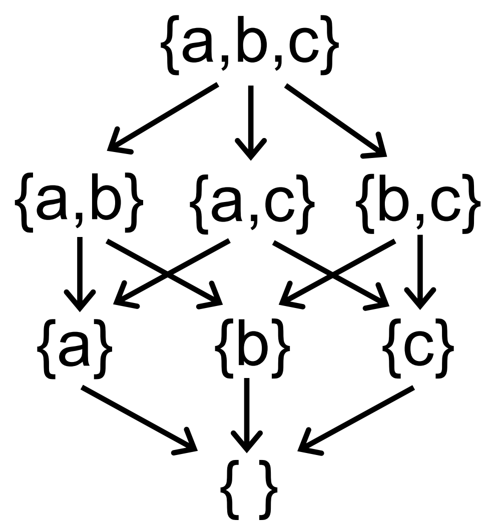

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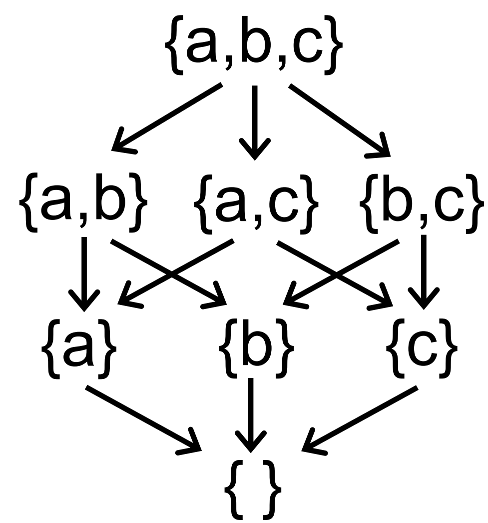

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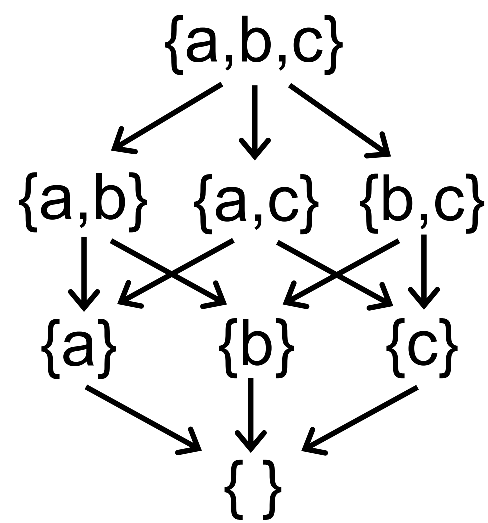

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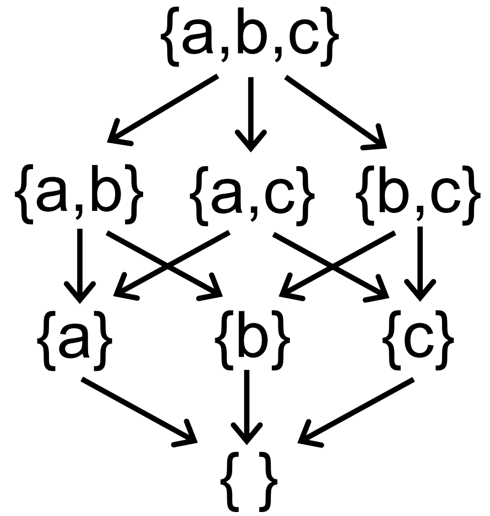

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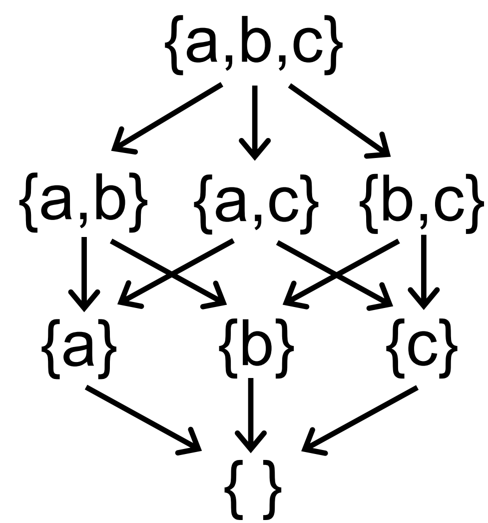

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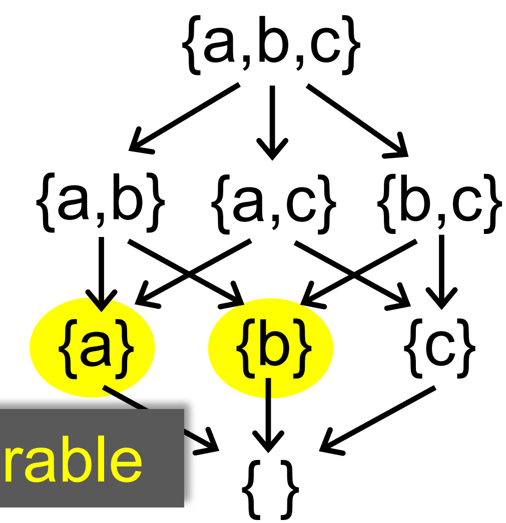

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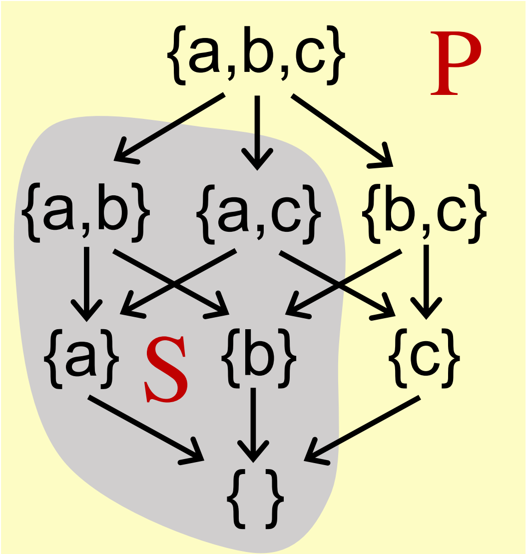

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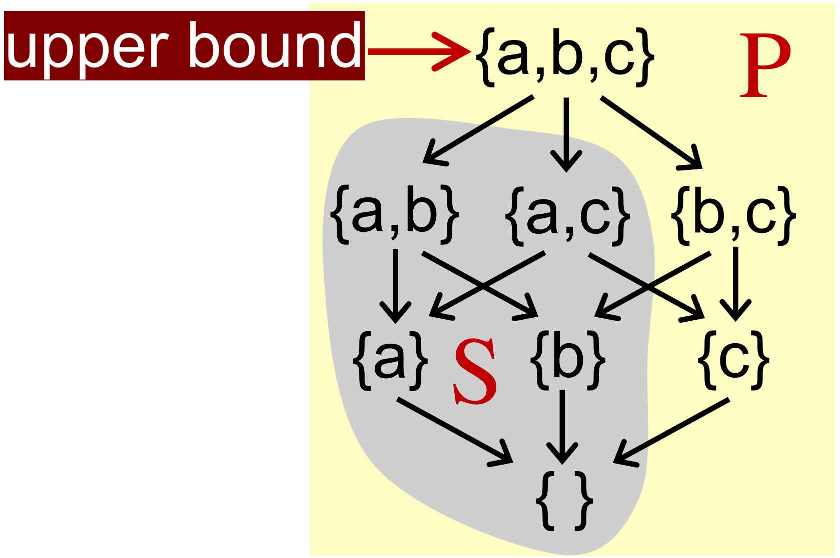

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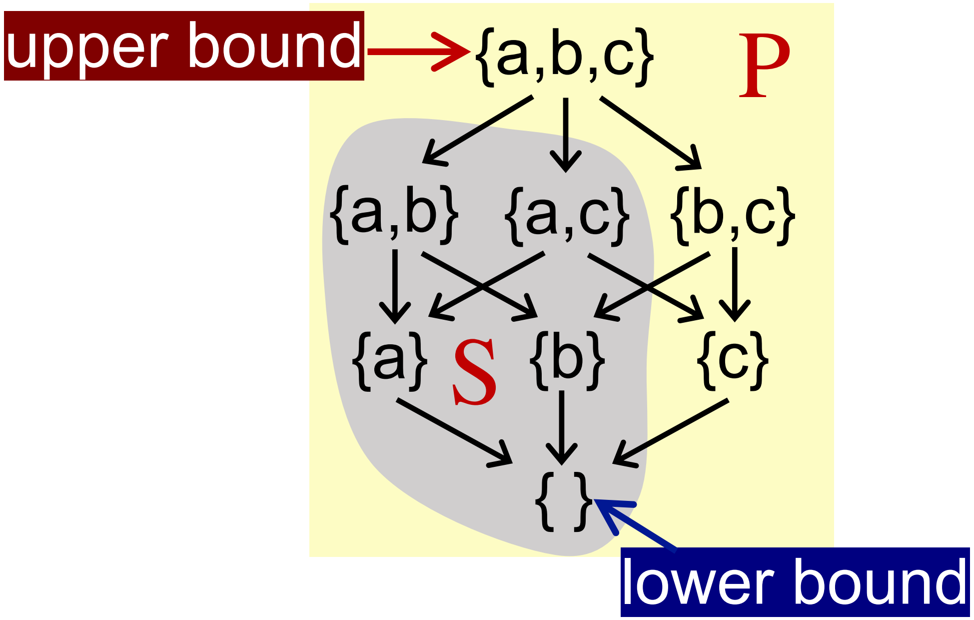

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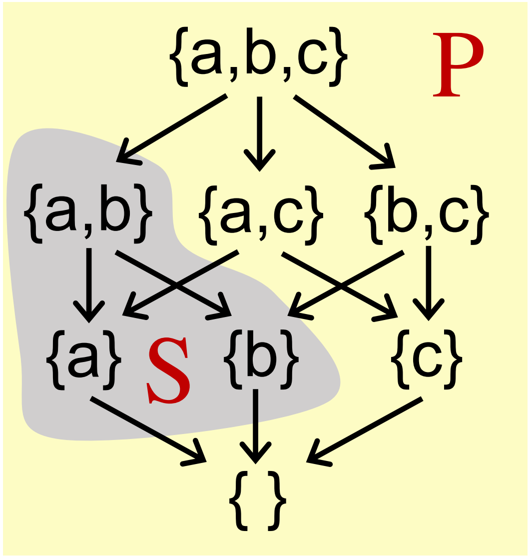

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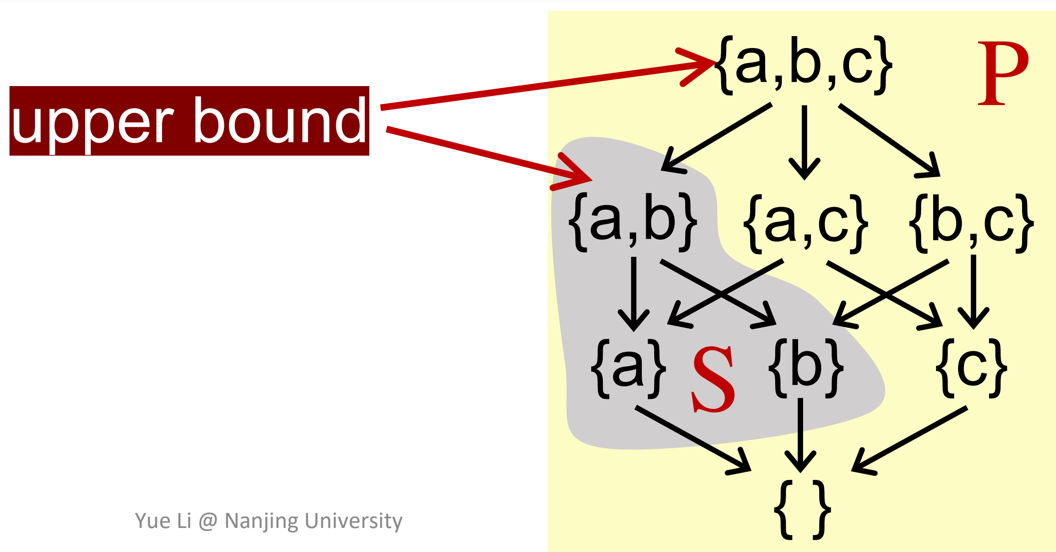

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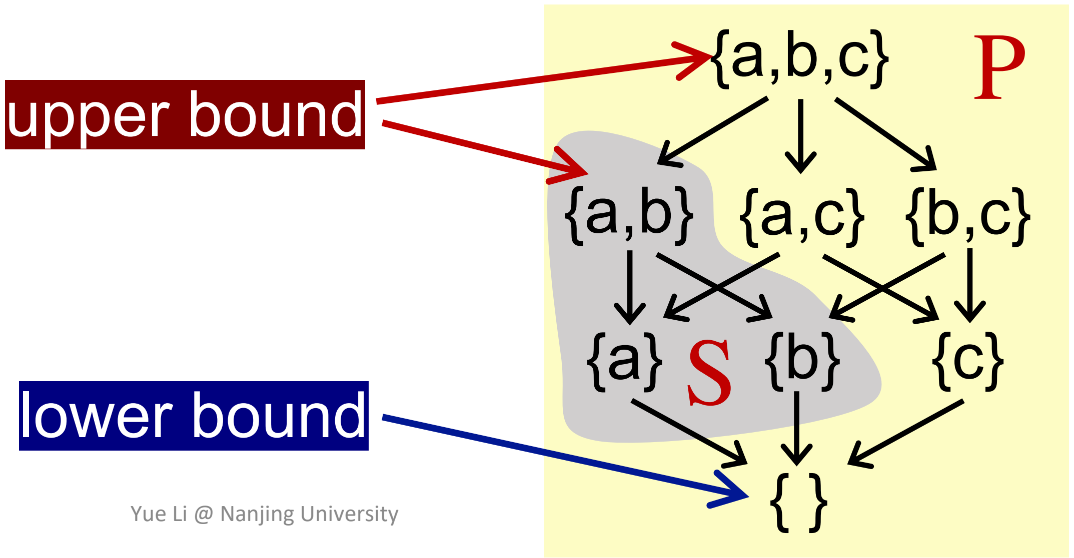

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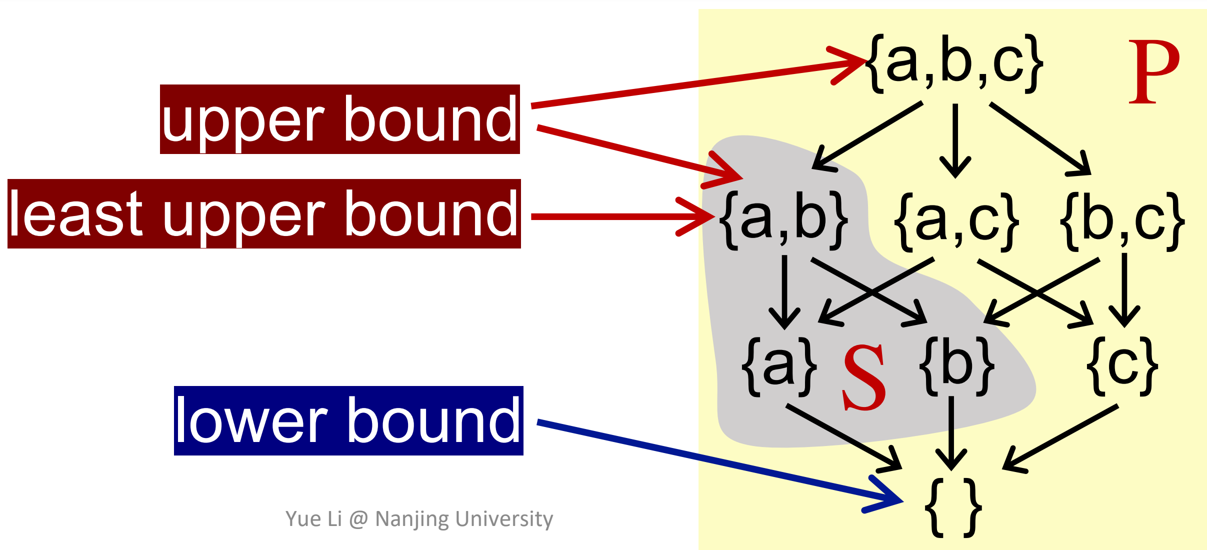

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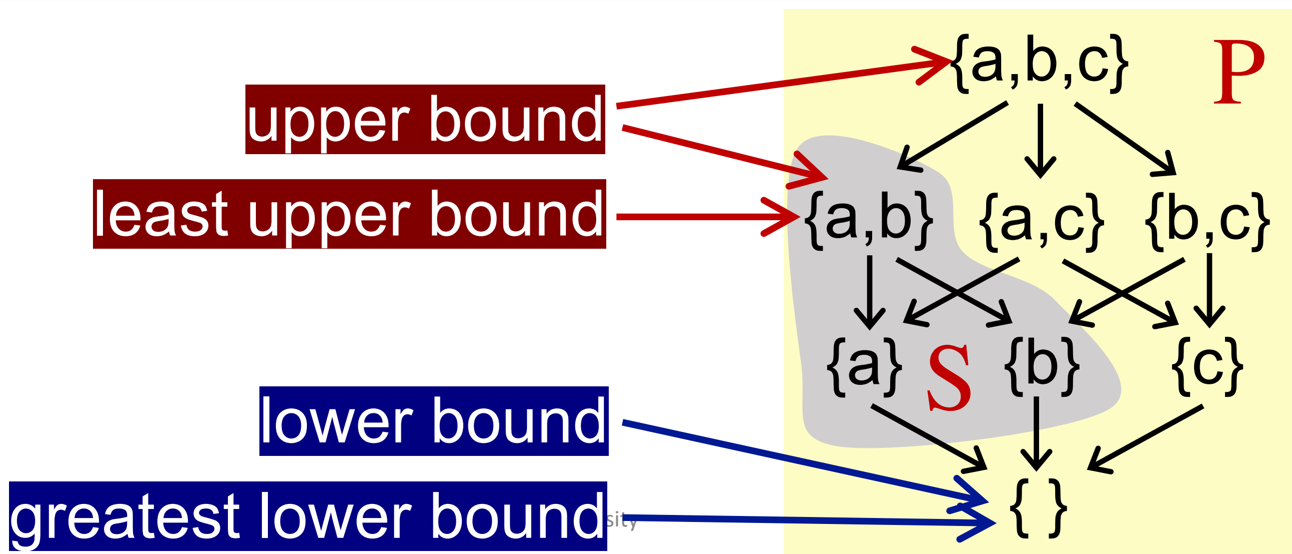

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Usually, if S contains only two elements a and b ($S = \{a, b\}$), then
$\sqcup S$ can be written $a \sqcup b$ (the join of a and b)
$\sqcap S$ can be written $a \sqcap b$ (the meet of a and b)

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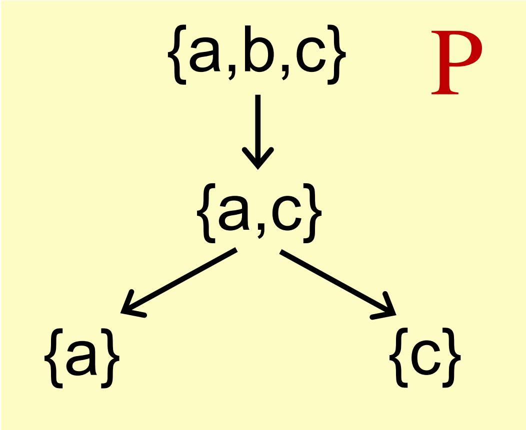

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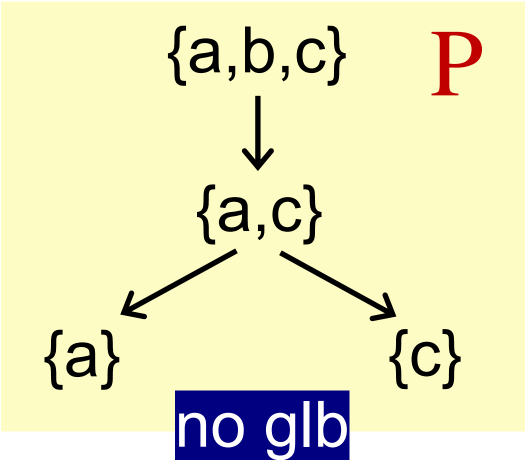

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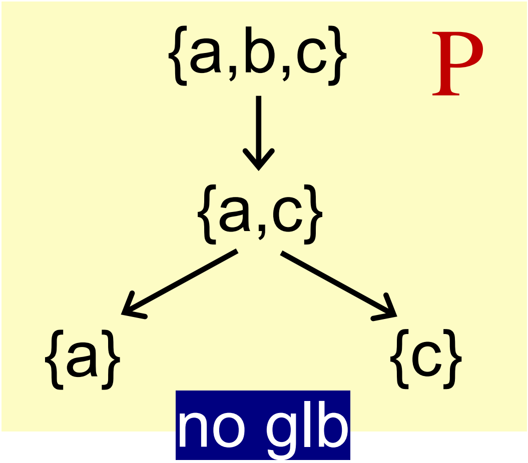

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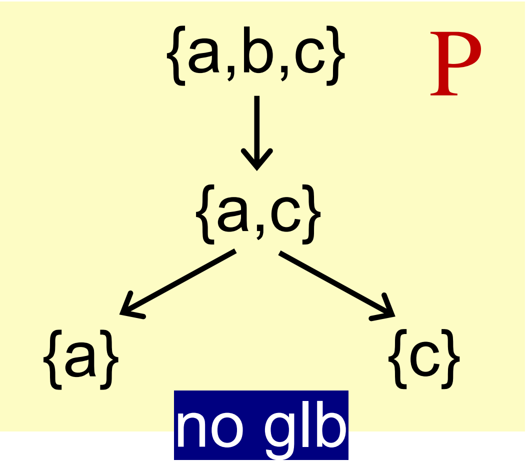

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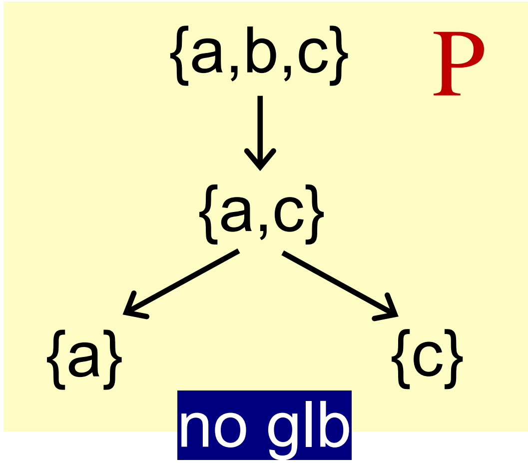

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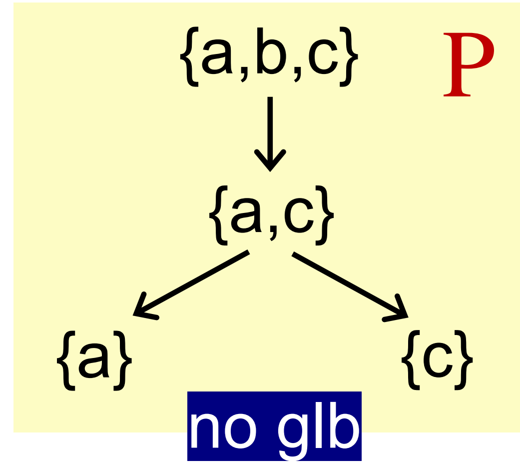

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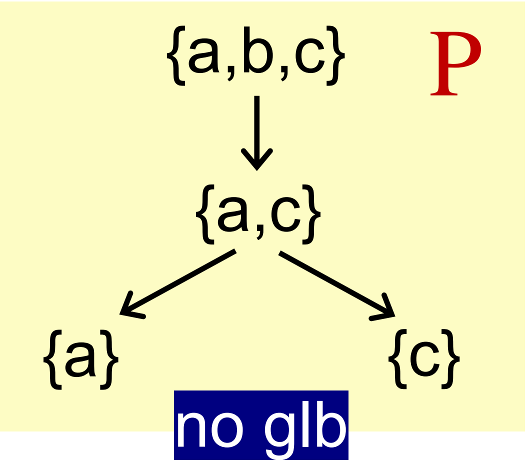

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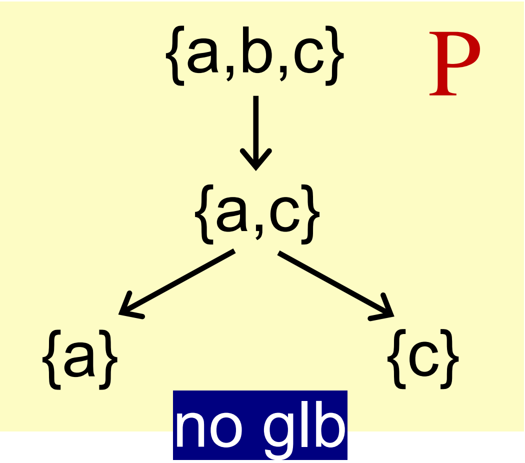

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by the *antisymmetry* of partial order $\sqsubseteq$

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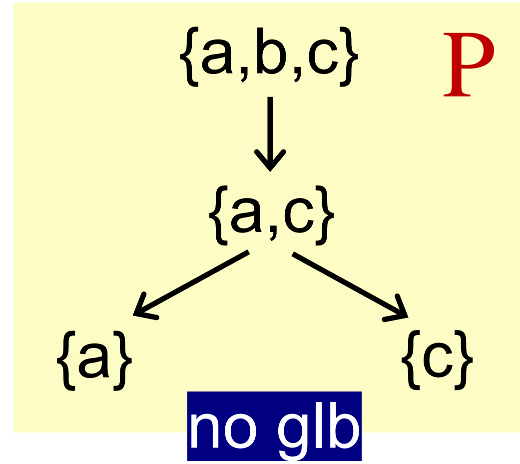

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according to the definition of glb

$g_1 \sqsubseteq (g_2 = \sqcap P)$ and $g_2 \sqsubseteq (g_1 = \sqcap P)$

by the *antisymmetry* of partial order $\sqsubseteq$

$g_1 = g_2$

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✓ The ⊔ operator means "max" and ⊓ operator means "min"

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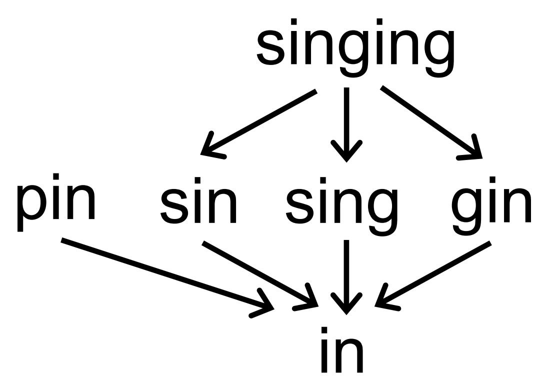

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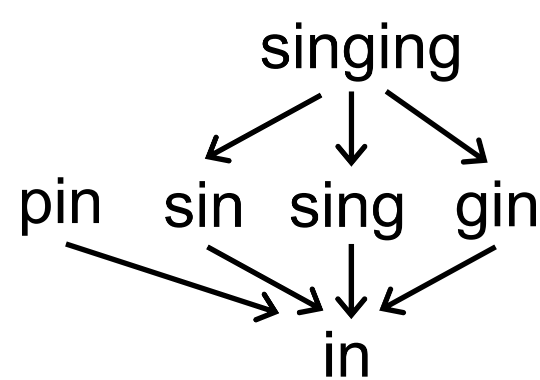

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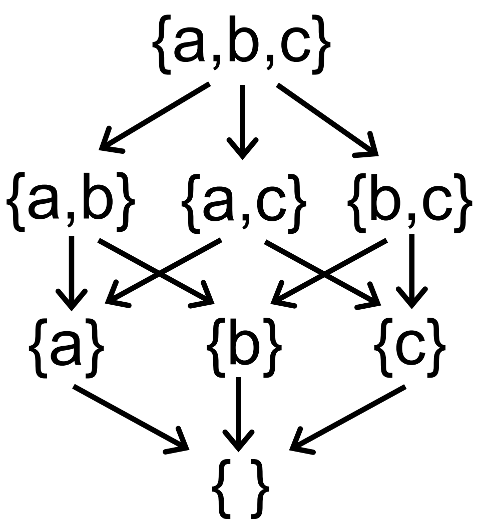

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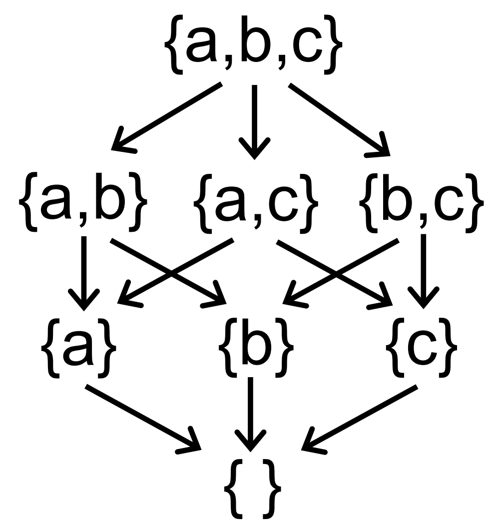

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Semilattice

Given a poset $(P, \sqsubseteq)$, $\forall a, b \in P$,
if only $a \sqcup b$ exists, then $(P, \sqsubseteq)$ is called a join semilattice
if only $a \sqcap b$ exists, then $(P, \sqsubseteq)$ is called a meet semilattice

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Given a lattice $(P, \sqsubseteq)$, for arbitrary subset S of P, if $\sqcup S$ and $\sqcap S$ exist, then $(P, \sqsubseteq)$ is called a complete lattice

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✗ For a subset S^+ including all positive integers, it has no $\sqcup S^+$ $(+\infty)$

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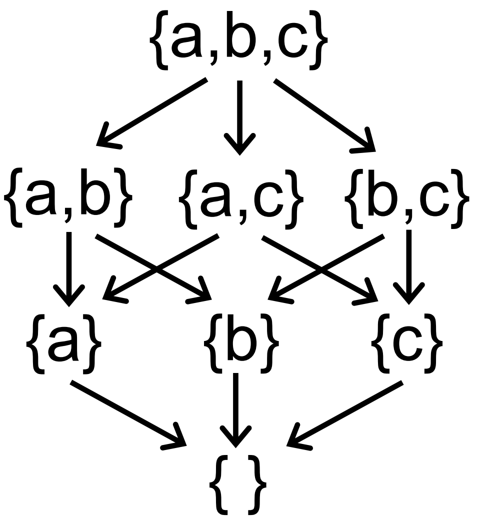

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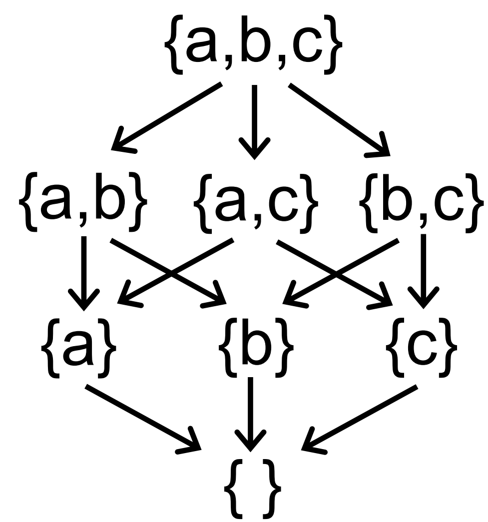
Note: the definition of bounds implies that the bounds are not necessarily in the subsets (but they must be in the lattice)



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a greatest element ⊤ = ⊔P called top and
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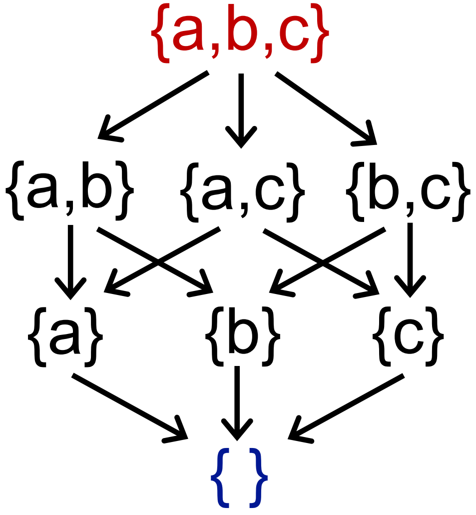

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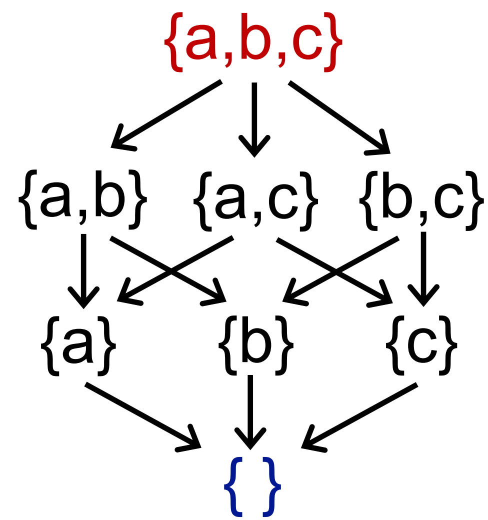

Complete Lattice Mostly focused in data flow analysis

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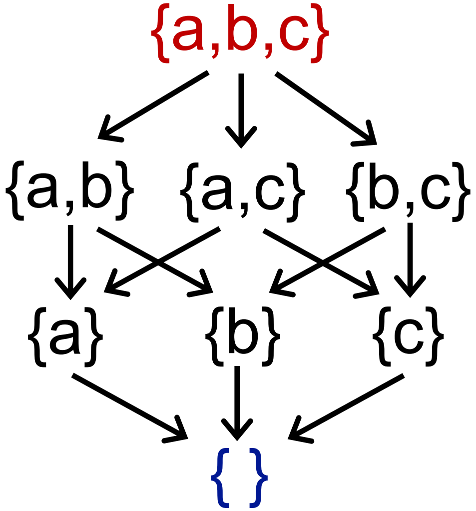

Product Lattice

Given lattices $L_1 = (P_1, \sqsubseteq_1)$, $L_2 = (P_2, \sqsubseteq_2)$, …, $L_n = (P_n, \sqsubseteq_n)$, if for all i, $(P_i, \sqsubseteq_i)$ has $\sqcup_i$ (least upper bound) and $\sqcap_i$ (greatest lower bound), then we can have a product lattice $L^n = (P, \sqsubseteq)$ that is defined by:

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- A product lattice is a lattice
- If a product lattice L is a product of complete lattices, then L is also complete

Data Flow Analysis Framework via Lattice

A data flow analysis framework (D, L, F) consists of:

- **D**: a direction of data flow: forwards or backwards

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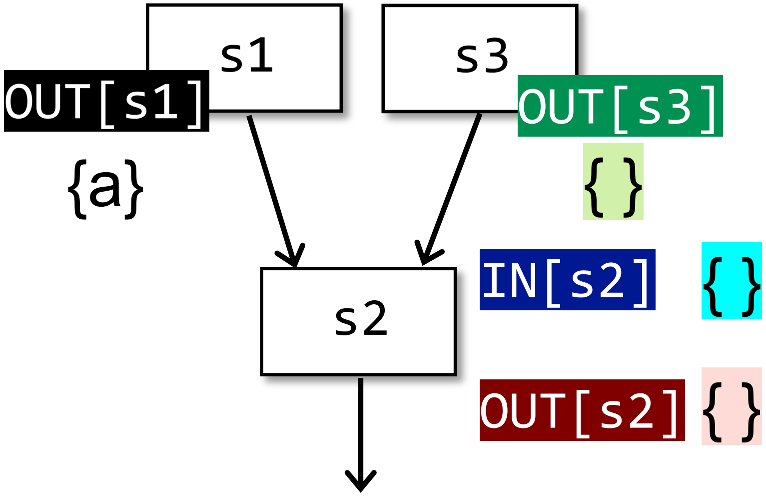
- **F**: a family of transfer functions from V to V

Data Flow Analysis Framework via Lattice



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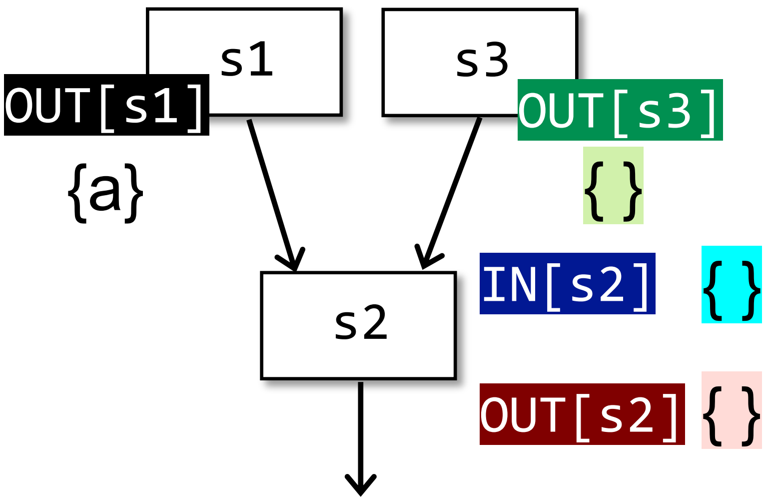
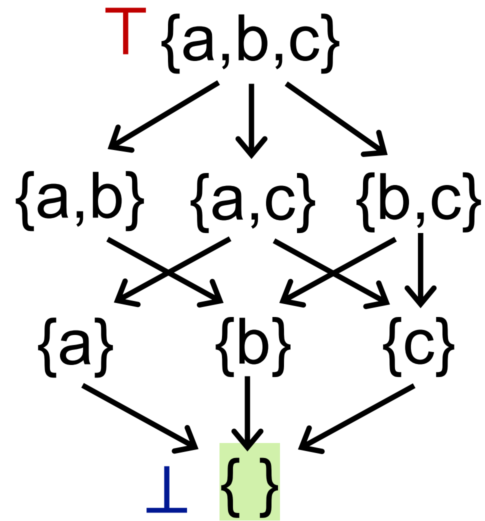
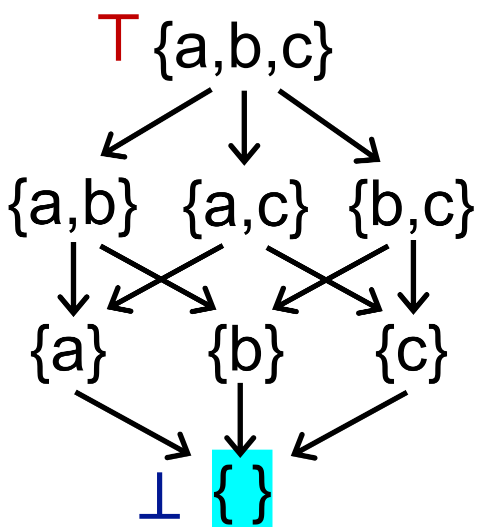
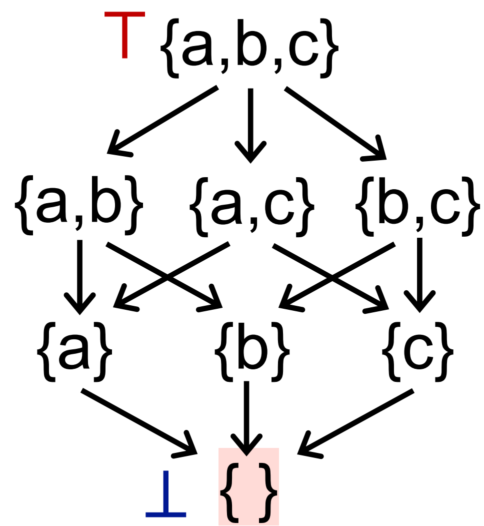




OUT[s3]'s Lattice



IN[s2]'s Lattice



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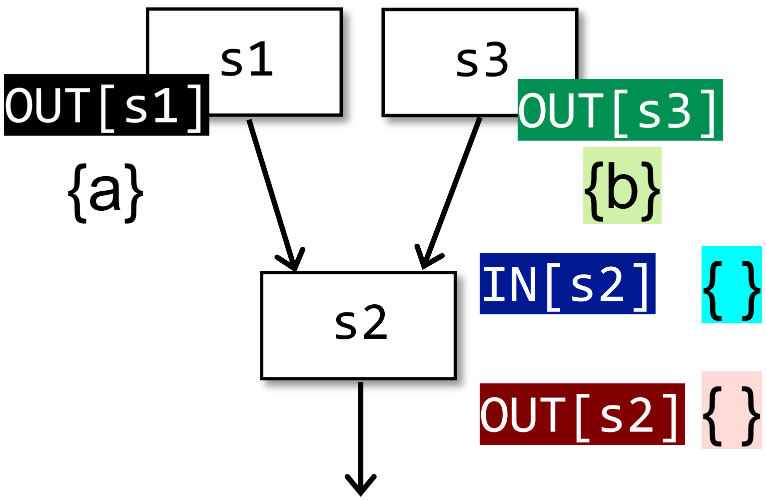

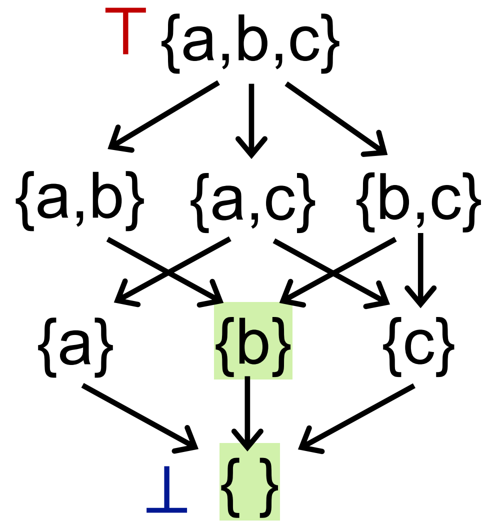

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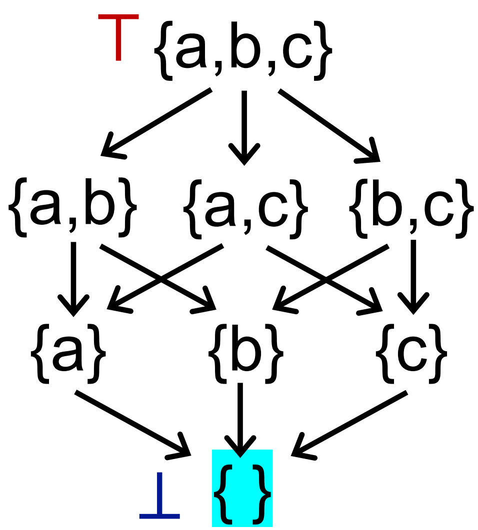

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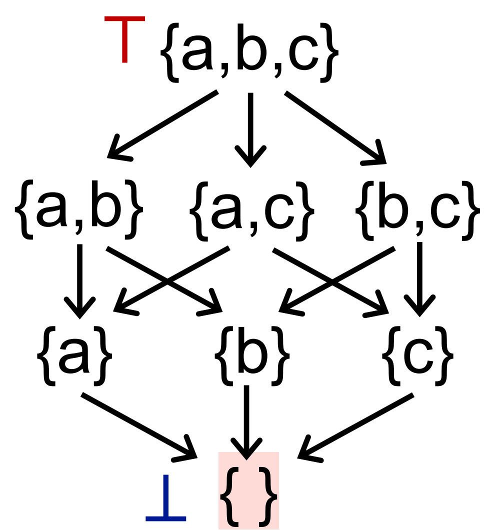
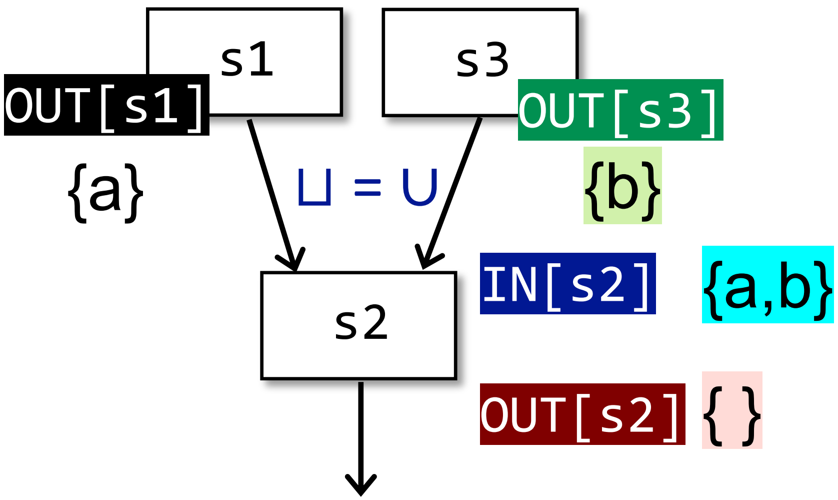
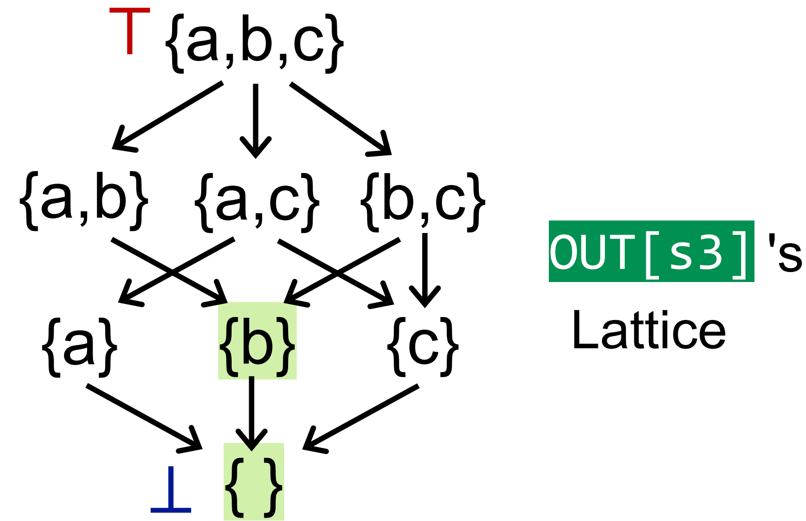
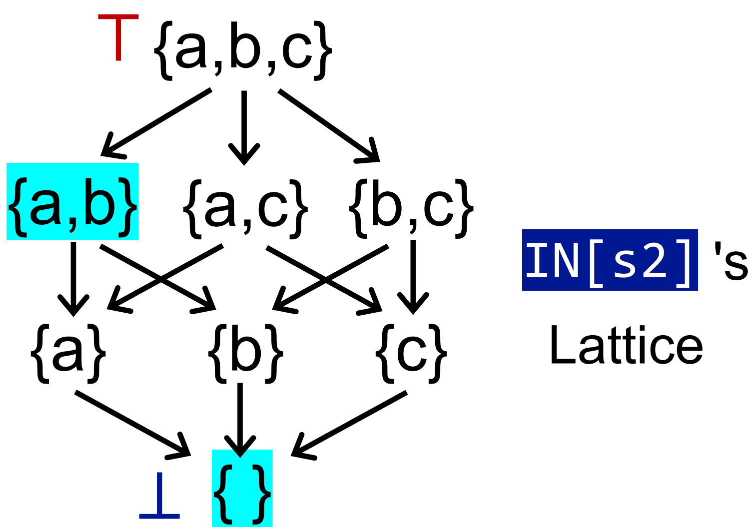


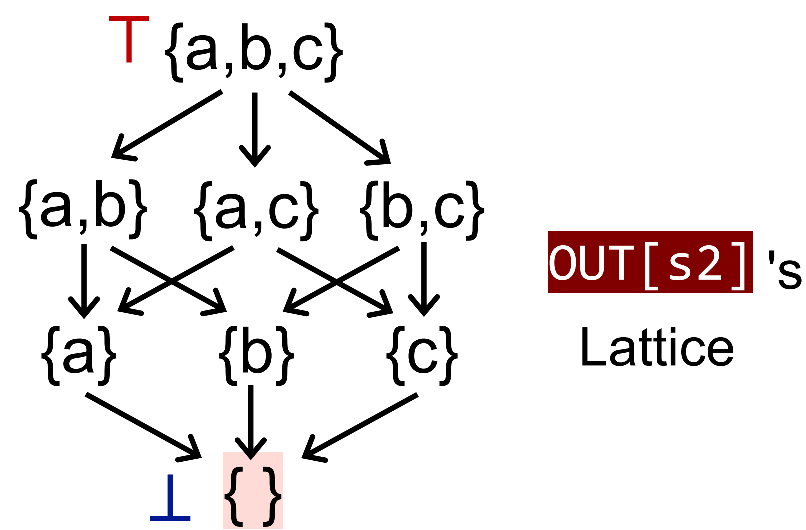
OUT[s2]'s Lattice

Data Flow Analysis Framework via Lattice









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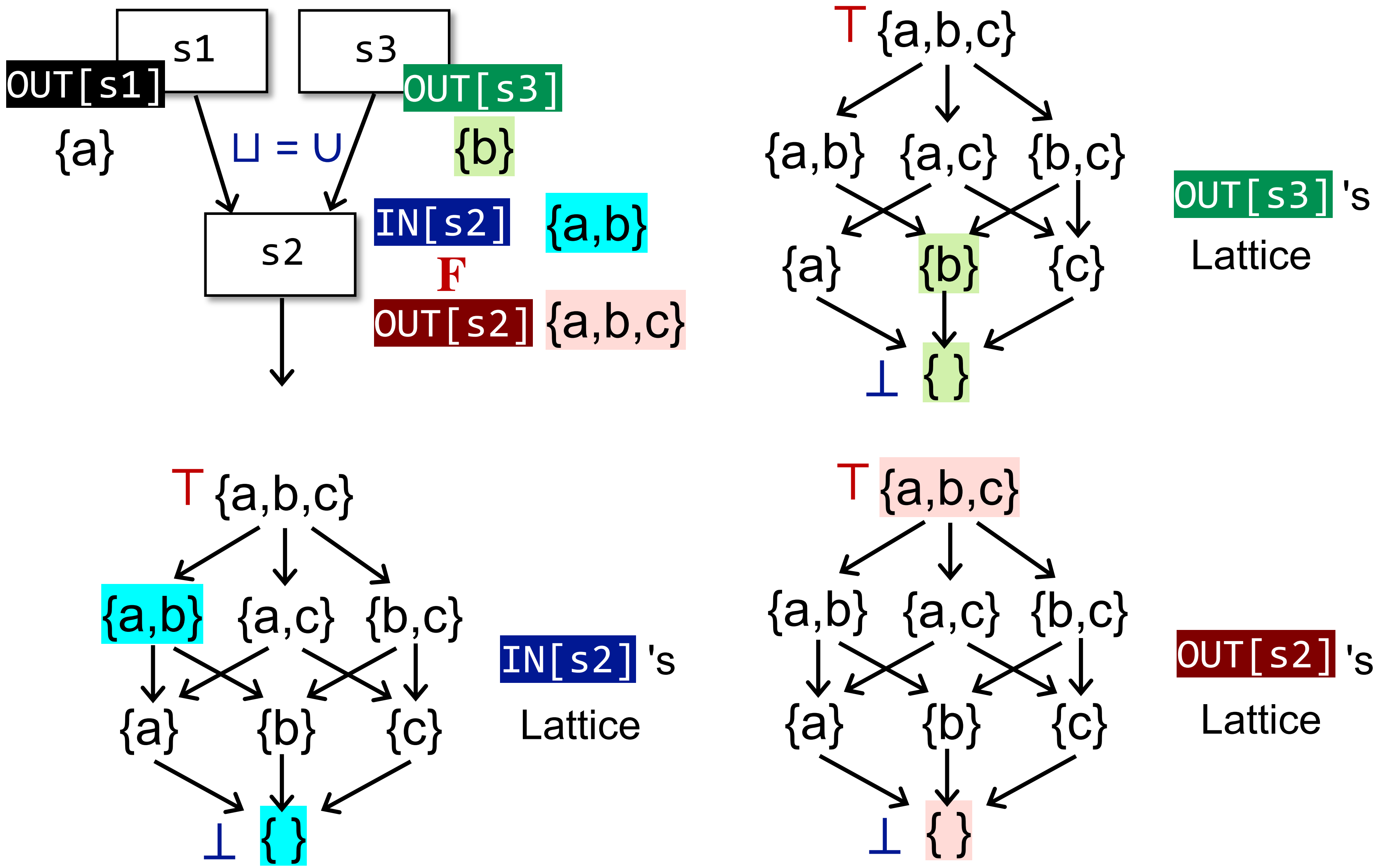

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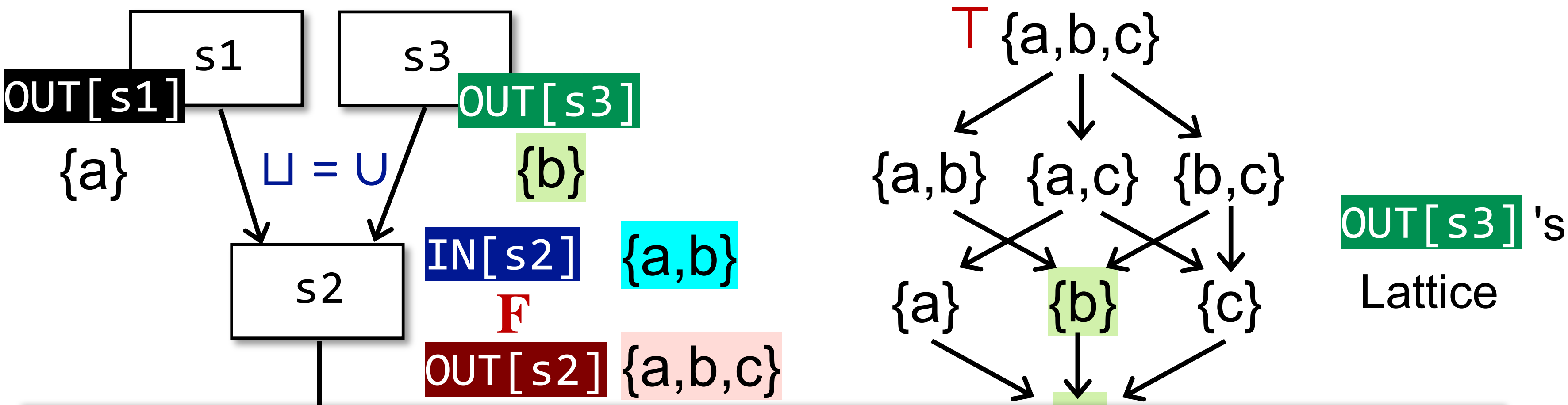
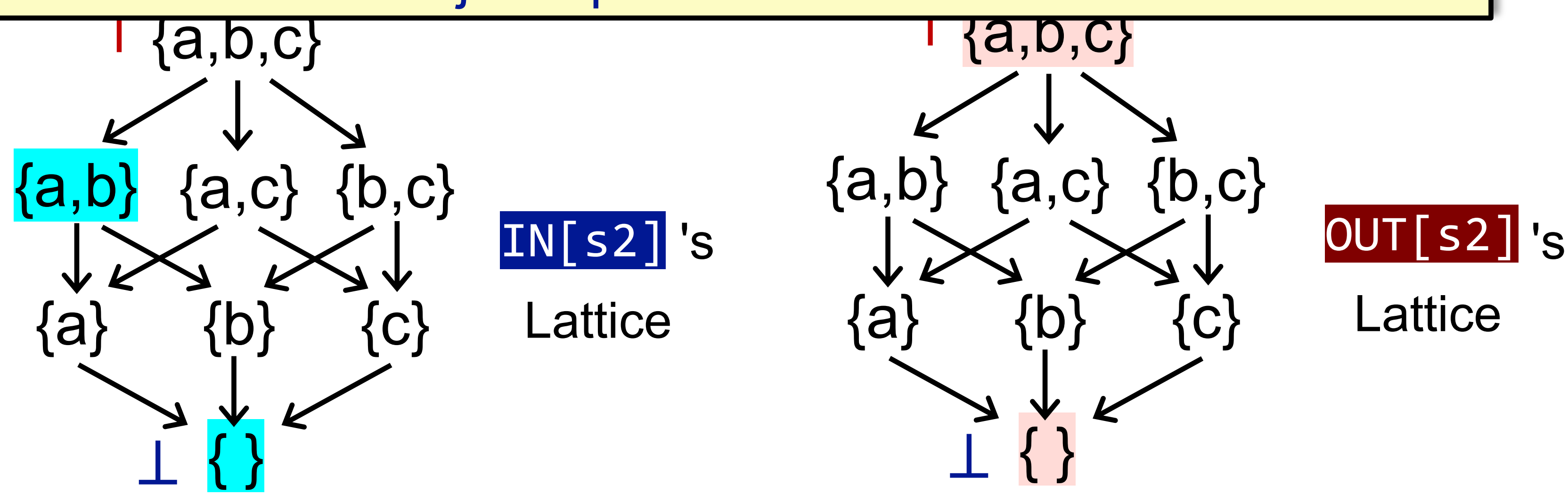


Data flow analysis can be seen as iteratively applying transfer functions and meet/join operations on the values of a lattice



Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
- When will the algorithm reach the fixed point, or when can we get the solution?

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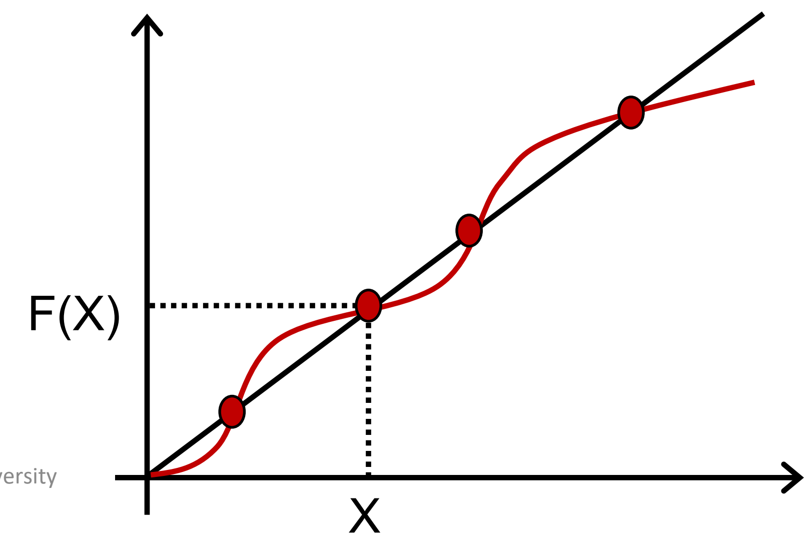

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Let us prove

(1) Existence of fixed point
(2) The fixed point is the least

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of $\perp$ and f: L $\rightarrow$ L, we have

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As f is monotonic, we have

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$$f(\bot) \sqsubseteq f(f(\bot)) = f^2(\bot)$$

By repeatedly applying f, we have an ascending chain

$$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ \ldots \sqsubseteq f^i(\bot)$$

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of $\perp$ and f: $L \rightarrow L$, we have

$$\perp \sqsubseteq f(\perp)$$

As f is monotonic, we have

$$f(\perp) \sqsubseteq f(f(\perp)) = f^2(\perp)$$

By repeatedly applying f, we have an ascending chain

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ \ldots \sqsubseteq f^i(\perp)$$

As L is finite (its height is H), the values are bounded among

$$\perp, f(\perp), f^2(\perp) \ldots f^H(\perp)$$

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When $i > H$, by pigeonhole principle, there exists k and j that

$$f^k(\perp) = f^j(\perp) \text{ (assume } k < j \leq H+1)$$

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Further as $f^k(\perp) \sqsubseteq \ldots \sqsubseteq f^j(\perp)$ (monotonicity of f), we have

$$f^{Fix} = f^k(\perp) = f^{k+1}(\perp) = \ldots = f^j(\perp) \text{ (proof by contradiction)}$$

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Thus, the fixed point exists.

Fixed-Point Theorem (Least Fixed Point)

Proof:
Assume we have another fixed point x, i.e., x = f(x)

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\perp$, we have $\perp \sqsubseteq x$

Fixed-Point Theorem (Least Fixed Point)

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Assume we have another fixed point x, i.e., x = f(x)
By the definition of ⊥, we have ⊥ ⊑ x
Induction begins:

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As f is monotonic, we have

$$f(\perp) \sqsubseteq f(x)$$

Fixed-Point Theorem (Least Fixed Point)

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Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\bot$, we have $\bot \sqsubseteq x$

Induction begins:

As f is monotonic, we have

$$f(\bot) \sqsubseteq f(x)$$

Assume $f^i(\bot) \sqsubseteq f^i(x)$, as f is monotonic, we have

$$f^{i+1}(\bot) \sqsubseteq f^{i+1}(x)$$

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Assume we have another fixed point x, i.e., $x = f(x)$

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Assume $f^i(\perp) \sqsubseteq f^i(x)$, as f is monotonic, we have

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Thus by induction, we have

$$f^i(\perp) \sqsubseteq f^i(x)$$

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Thus by induction, we have

$$f^i(\bot) \sqsubseteq f^i(x)$$

Thus $f^i(\bot) \sqsubseteq f^i(x) = x$ (as x is a fixed point regardless of i), then we have

$$f^{Fix} = f^k(\bot) \sqsubseteq f^k(x) = x$$

Thus the fixed point is the least

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\perp$, we have $\perp \sqsubseteq x$

Induction begins:

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The proof for greatest fixed point is similar

Fixed-Point Theorem

Given a complete lattice $(L, \sqsubseteq)$, if

(1) $f: L \rightarrow L$ is monotonic and (2) L is finite, then

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$f(\perp), f(f(\perp)), \ldots, f^k(\perp)$ until a fixed point is reached

the greatest fixed point of f can be found by iterating

$f(\top), f(f(\top)), \ldots, f^k(\top)$ until a fixed point is reached

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- ? Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- ? If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
- When will the algorithm reach the fixed point, or when can we get the solution?

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- When will the algorithm reach the fixed p... can we get the solution?

greatest or least fixed point

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greatest or least fixed point

> Now what we have just seen is the property (fixed point theorem) for the function on a lattice. We cannot say our iterative algorithm also has that property unless we can *relate the algorithm to the fixed point theorem*, if possible

Contents (1)

1. Iterative Algorithm, Another View
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3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem

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7. Relate Iterative Algorithm to Fixed Point Theorem

8. May/Must Analysis, A Lattice View

9. MOP and Distributivity

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Static Program Analysis

Data Flow Analysis — Foundations

Nanjing University

Yue Li

2022

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Relate Iterative Algorithm to Fixed-Point Theorem

$\rightarrow (\bot,\ \bot,\ \ldots,\ \bot)$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

$\Downarrow$

Given a complete lattice $(L, \sqsubseteq)$, if

(1) $f: L \rightarrow L$ is monotonic and (2) L is finite, then

the least fixed point of f can be found by iterating

$f(\bot), f(f(\bot)), \ldots, f^k(\bot)$ until a fixed point is reached

the greatest fixed point of f can be found by iterating

$f(\top), f(f(\top)), \ldots, f^k(\top)$ until a fixed point is reached

Relate Iterative Algorithm to Fixed-Point Theorem

(L, L, …, L)

If a product lattice L^k is a product of complete (and finite) lattices, i.e., (L, L, …, L), then L^k is also complete (and finite)

$\rightarrow (\bot, \bot, \ldots, \bot)$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

$\Downarrow$

Given a complete lattice (L, ⊑), if
(1) f: L → L is monotonic and $\boxed{\text{(2) L is finite,}}$ then
the least fixed point of f can be found by iterating
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If a product lattice L^k is a product of complete (and finite) lattices, i.e., (L, L, …, L), then L^k is also complete (and finite)

In each iteration, it is equivalent to think that we apply function F which consists of

(1) transfer function f_i: L → L for every node

(2) join/meet function ⊔/⊓: L×L → L for control-flow confluence

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Relate Iterative Algorithm to Fixed-Point Theorem

(L, L, …, L)

$\rightarrow (\perp, \ \perp, \ \dots, \ \perp)$

iter 1 $\rightarrow (v_1^1, v_2^1, \dots, v_k^1)$

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Given a complete lattice (L, ⊑), if
(1) f: L → L is monotonic and (2) L is finite, then
the least fixed point of f can be found by iterating
f(⊥), f(f(⊥)), …, f^k(⊥) until
the greatest fixed point of f can b
f(⊤), f(f(⊤)), …, f^k(⊤) until

Now the remaining issue is to prove that function F is monotonic

Prove Function F is Monotonic

In each iteration, it is equivalent to think that we apply function F which consists of

(1) transfer function f_i: $L \rightarrow L$ for every node

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Actually the binary operator is a basic case of $L \times L \times \ldots \times L$,

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Proof.
$\forall x, y, z \in L$, $x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

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Actually the binary operator is a basic case of L × L ×… × L,

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Proof.

$\forall x, y, z \in L, x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

by the definition of ⊔, $y \sqsubseteq y \sqcup z$

Prove Function F is Monotonic

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by transitivity of ⊑, $x \sqsubseteq y \sqcup z$

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by the definition of $\sqcup$, $y \sqsubseteq y \sqcup z$

by transitivity of $\sqsubseteq$, $x \sqsubseteq y \sqcup z$

thus $y \sqcup z$ is an upper bound for x, and also for z (by $\sqcup$'s definition)

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as $x \sqcup z$ is the least upper bound of x and z

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Thus the fixed point theorem applies to the iterative algorithm for data flow analysis

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- ? Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
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greatest or least fixed point

- When will the algorithm reach the fixed point, or when can we get the solution?

> Now what we have just seen is the property (fixed point theorem) for the function on a lattice. ~~We cannot say our iterative algorithm also has that property unless~~ we can relate the algorithm to the fixed point theorem, ~~if possible~~

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When Will the Algorithm Reach the Fixed Point?

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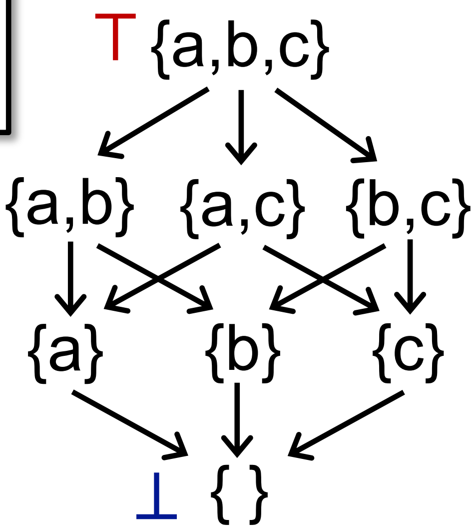
The height of a lattice *h* is the length of the longest path from Top to Bottom in the lattice.



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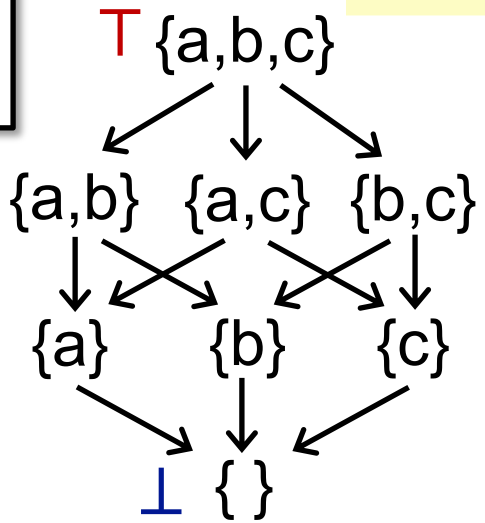
The height of a lattice h is the length of the longest path from Top to Bottom in the lattice.

$h = 3$



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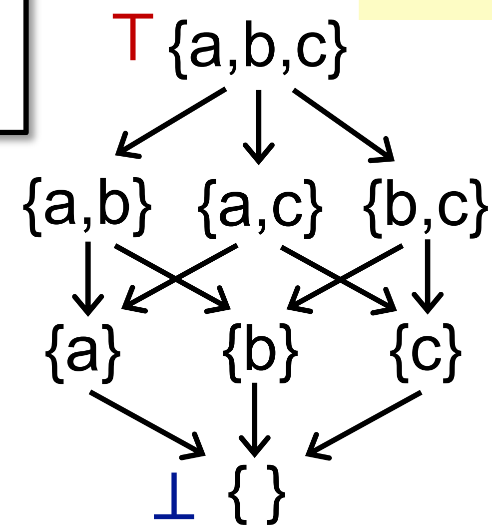

The maximum iterations i needed to reach the fixed point

$$(\perp, \ \perp, \ \ldots, \ \perp)$$

iter 1 $(v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $(v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $(v_1^i, v_2^i, \ldots, v_k^i)$

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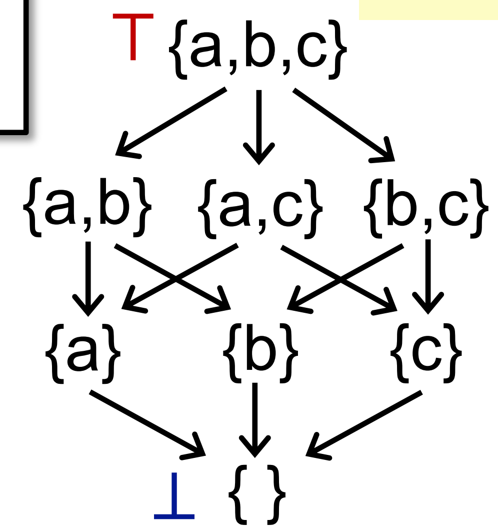

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iter i $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $(v_1^i, v_2^i, \ldots, v_k^i)$

In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

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The **height of a lattice h** is the length of the longest path from Top to Bottom in the lattice.

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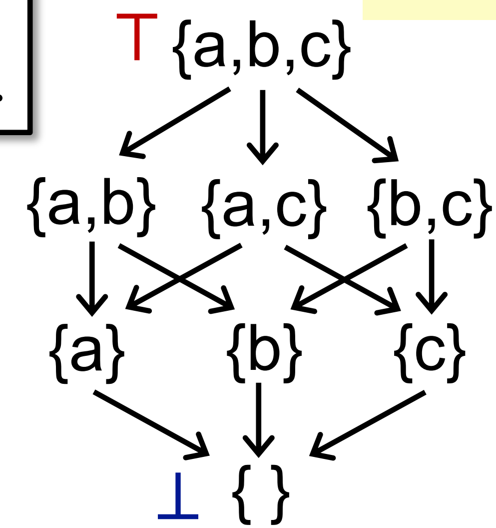

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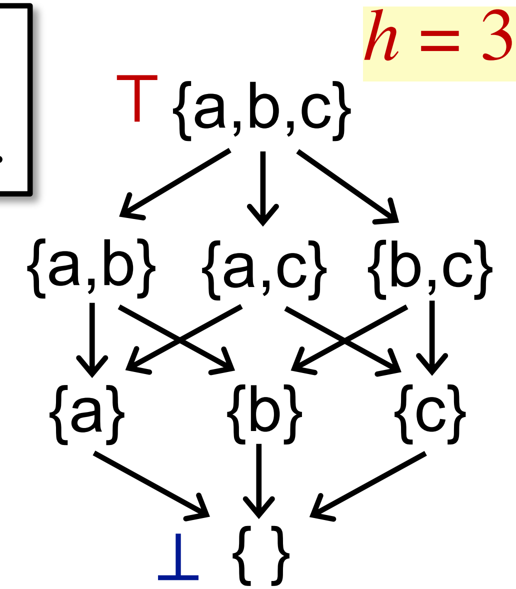
In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

Assume the lattice height is h and the number of nodes in CFG is k

When Will the Algorithm Reach the Fixed Point?

The height of a lattice h is the length of the longest path from Top to Bottom in the lattice.



The maximum iterations i needed to reach the fixed point

$$(\perp, \ \perp, \ \ldots, \ \perp)$$

iter 1 $(v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $(v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $(v_1^i, v_2^i, \ldots, v_k^i)$

In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

Assume the lattice height is h and the number of nodes in CFG is k

We need at most $i = h*k$ iterations

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

YES

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?

- If so, ~~is there only one solution or only one fixed point~~? If more than one, is our solution the best one (most precise)?

YES

- When will the algorithm reach the fixed point, or when can we get the solution?

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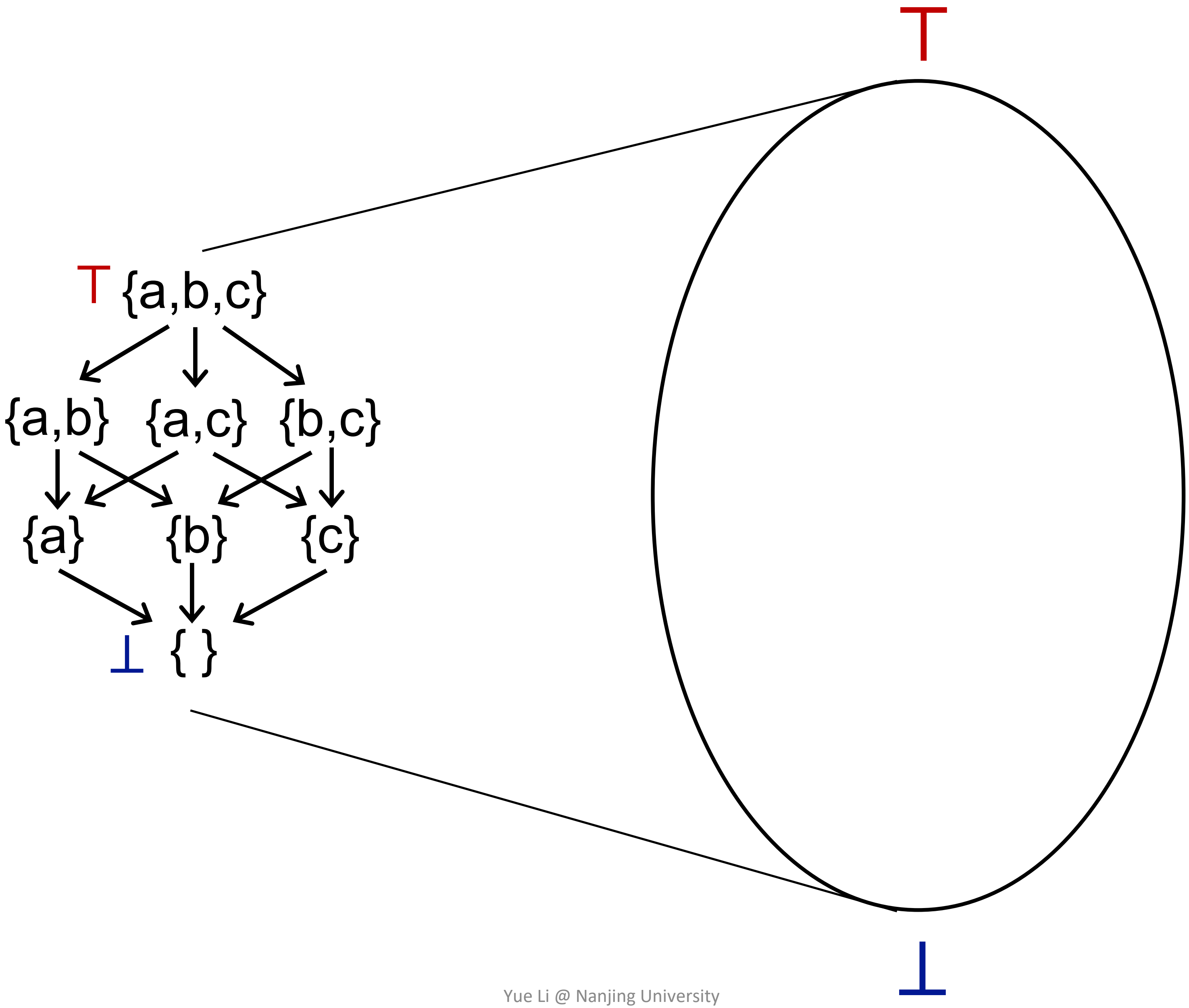
Worst case of #iterations:
the product of the lattice height and
the number of nodes in CFG

May and Must Analyses, a Lattice View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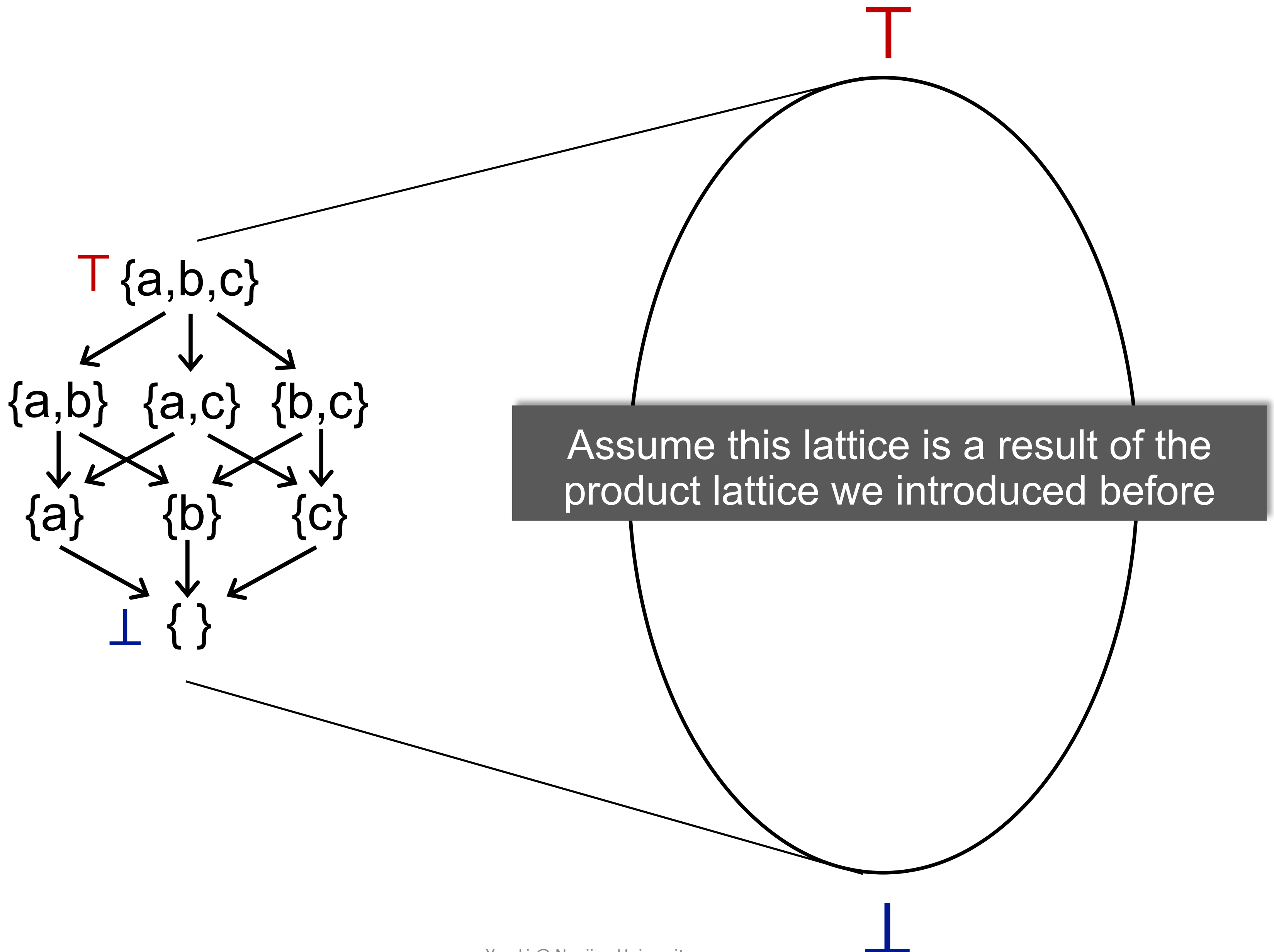
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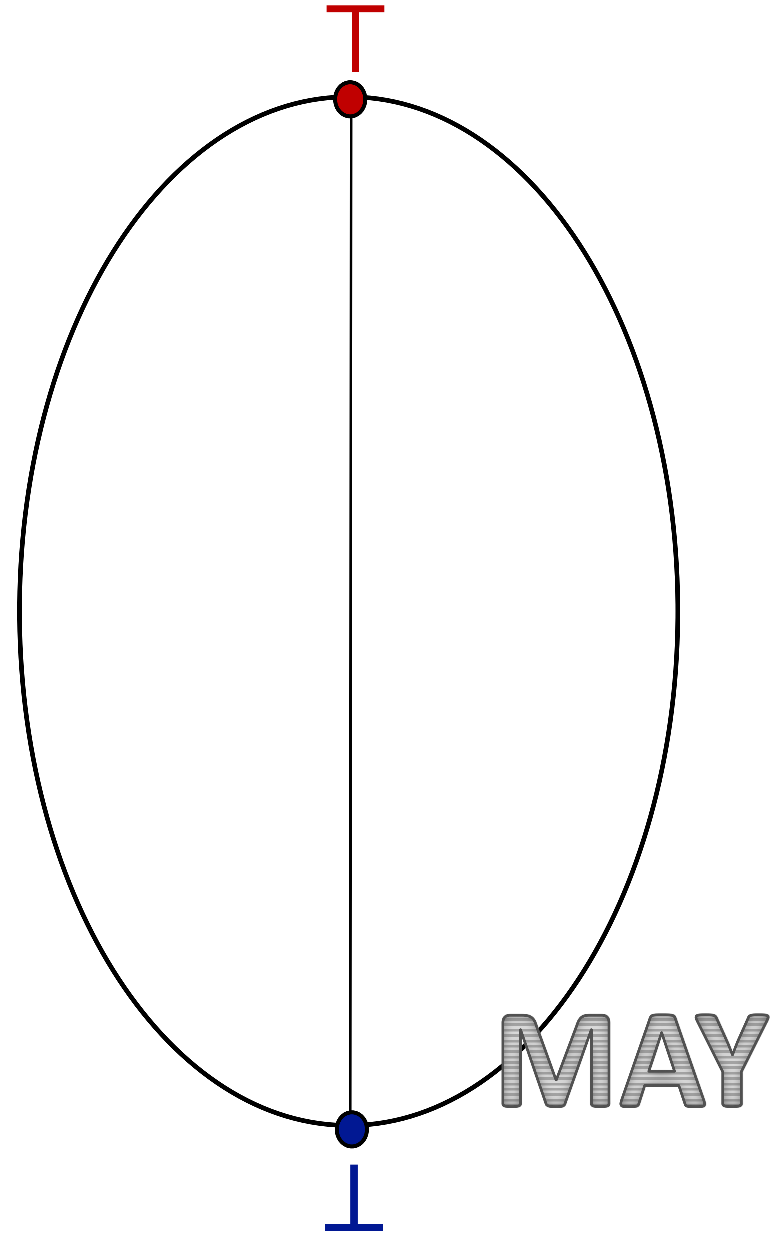
Assume this lattice is a result of the product lattice we introduced before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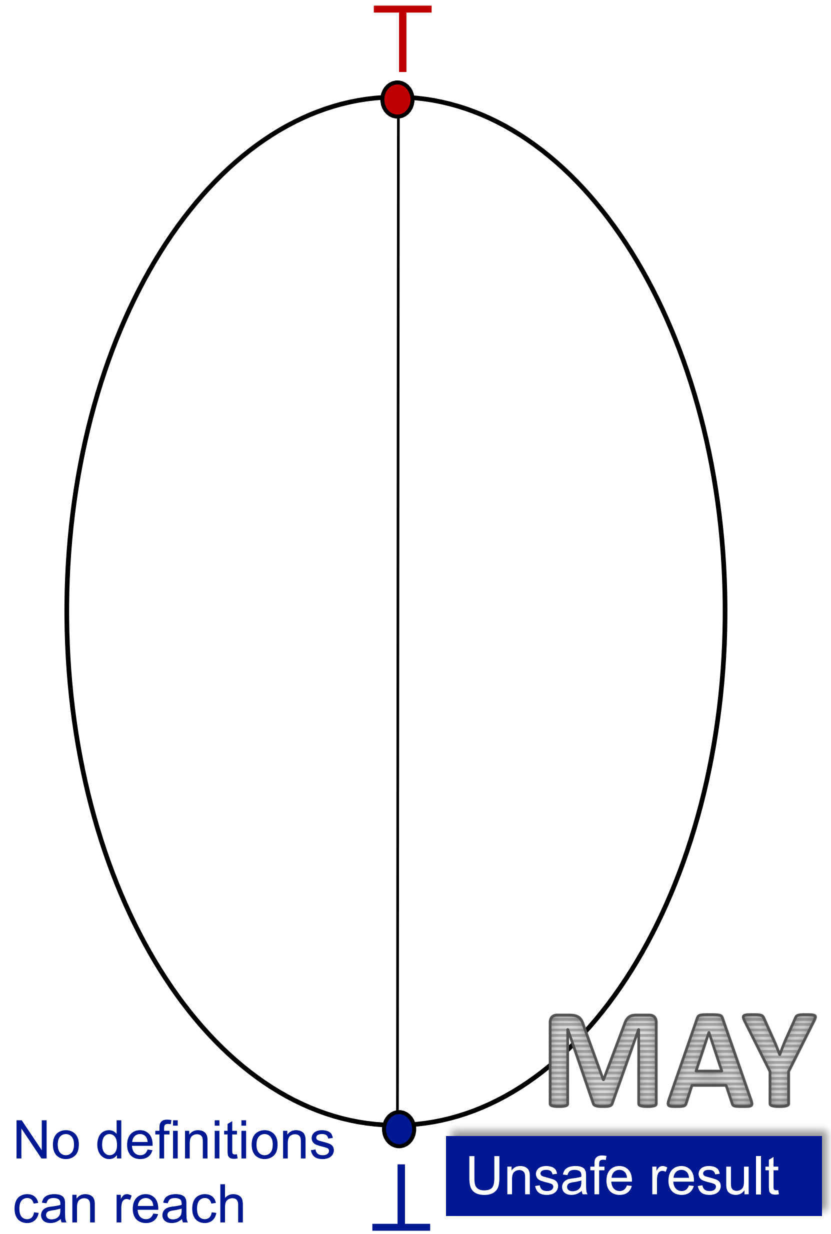
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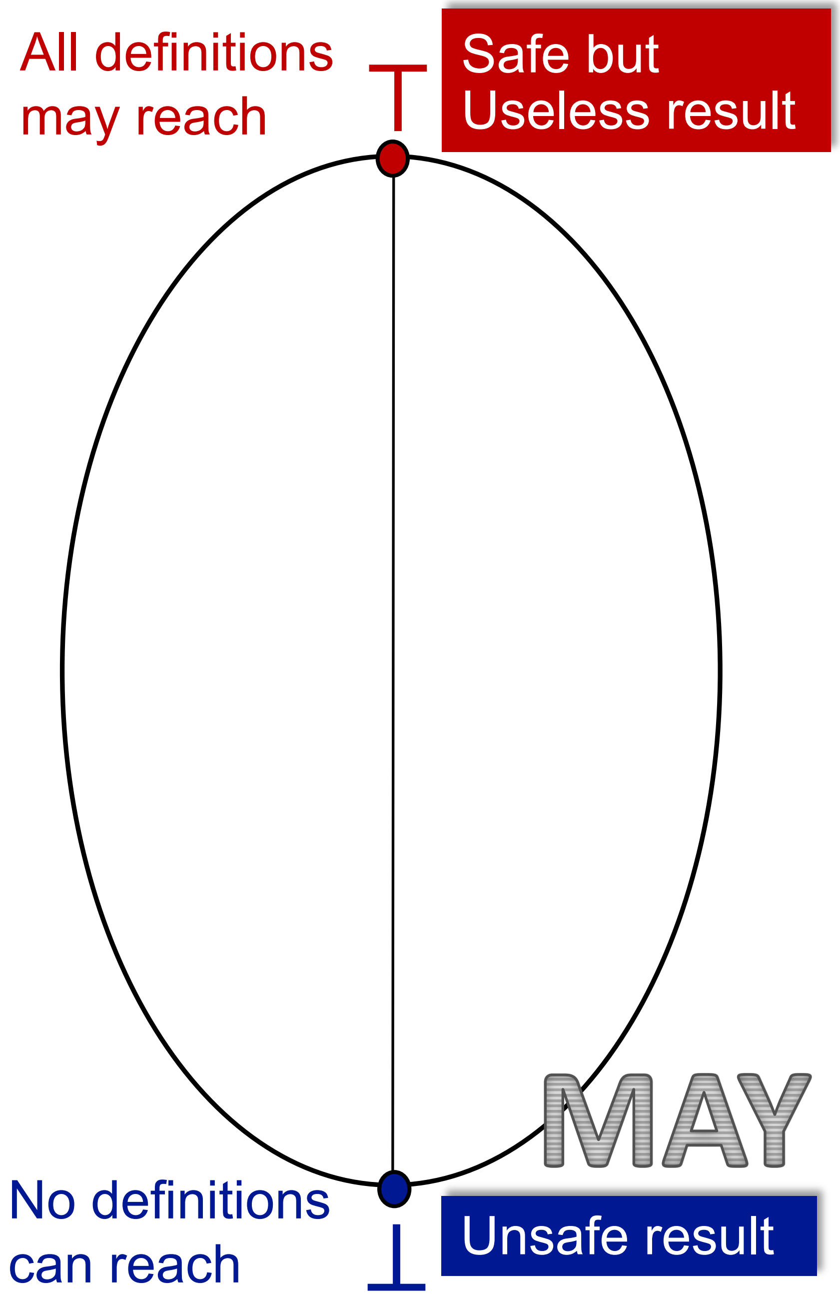
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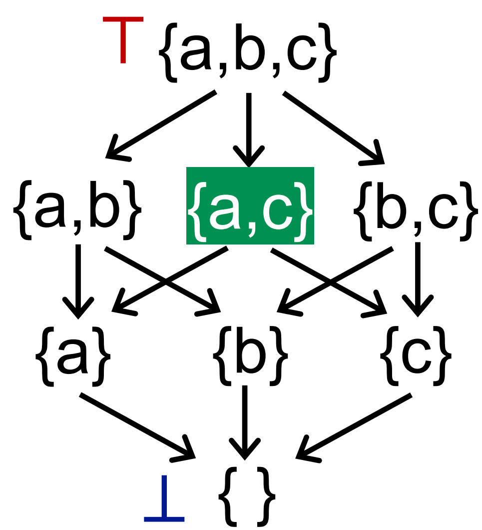
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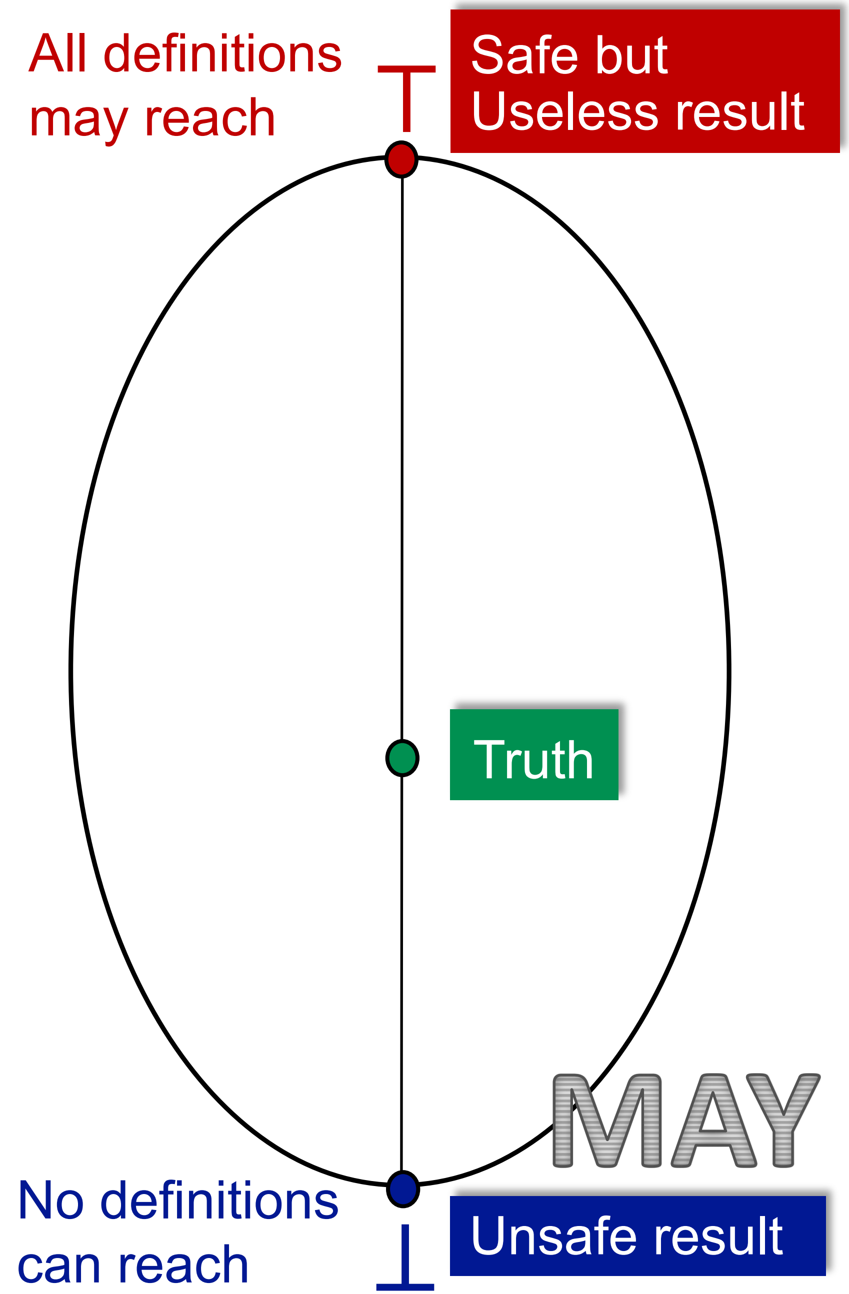
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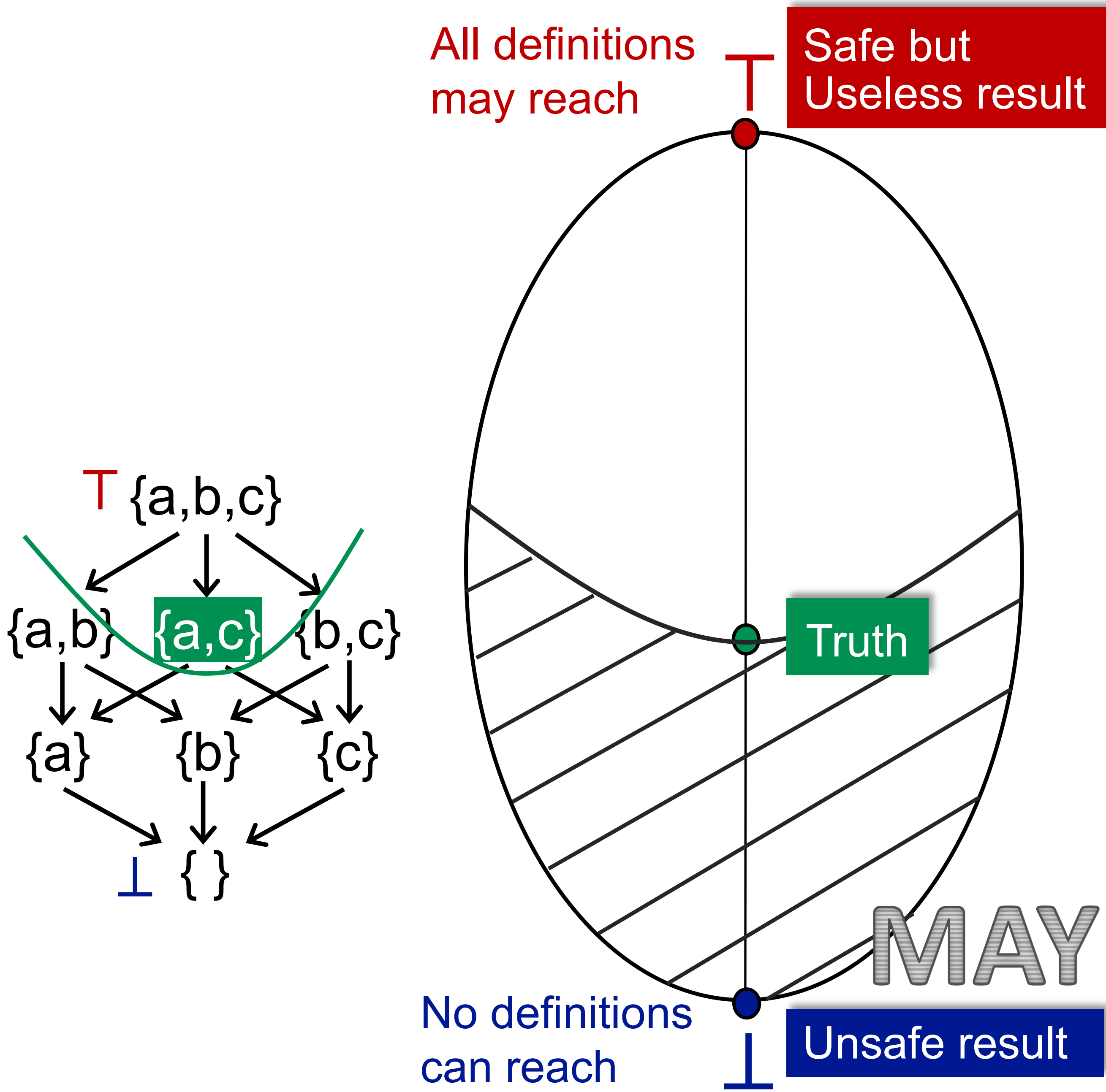
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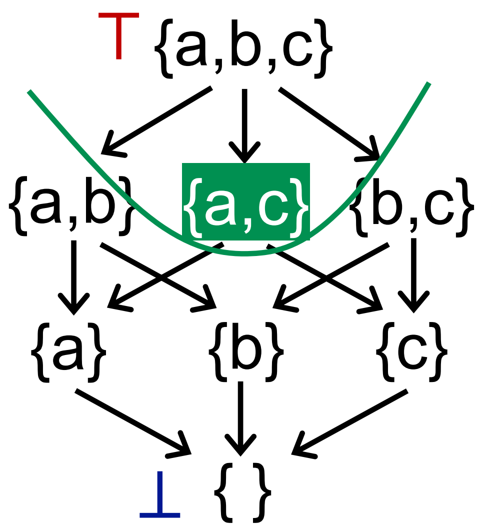

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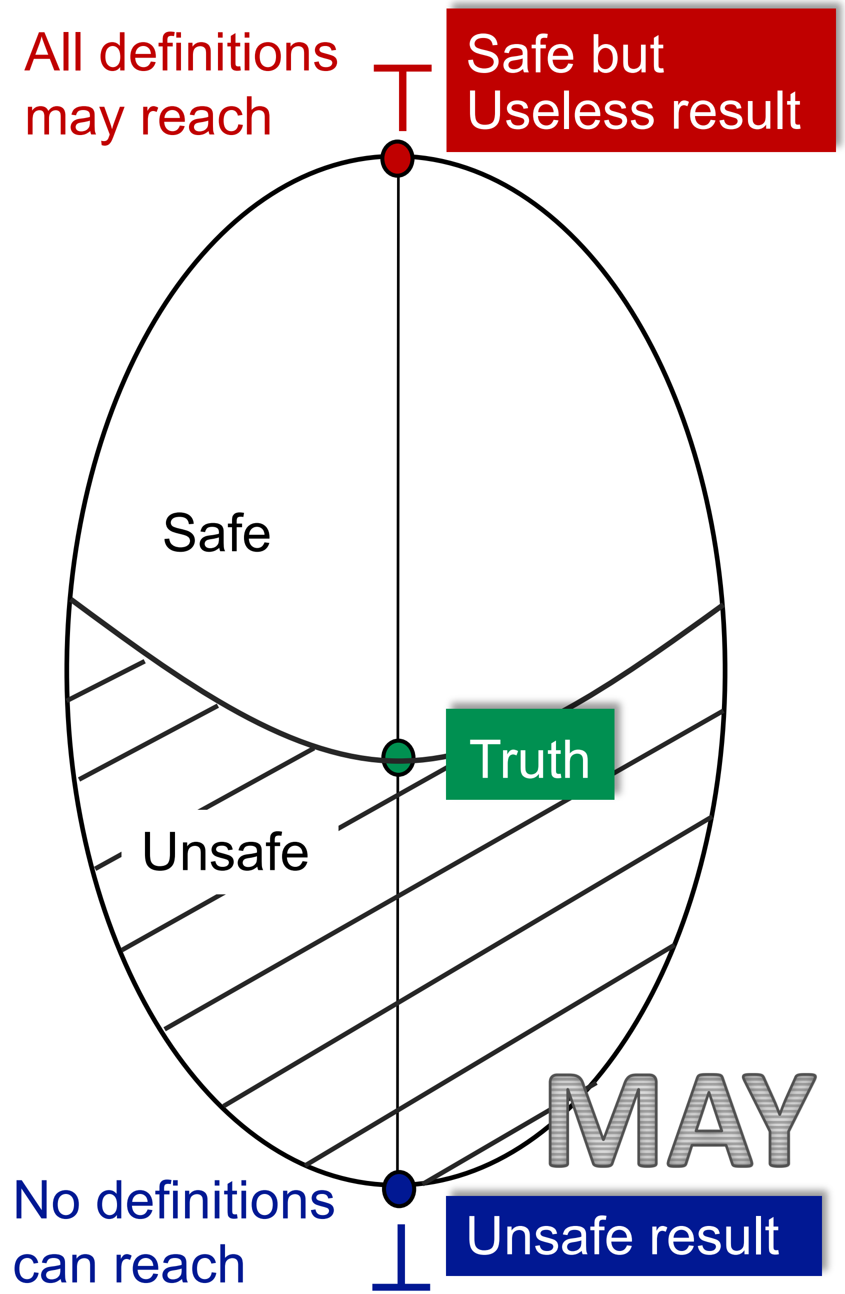

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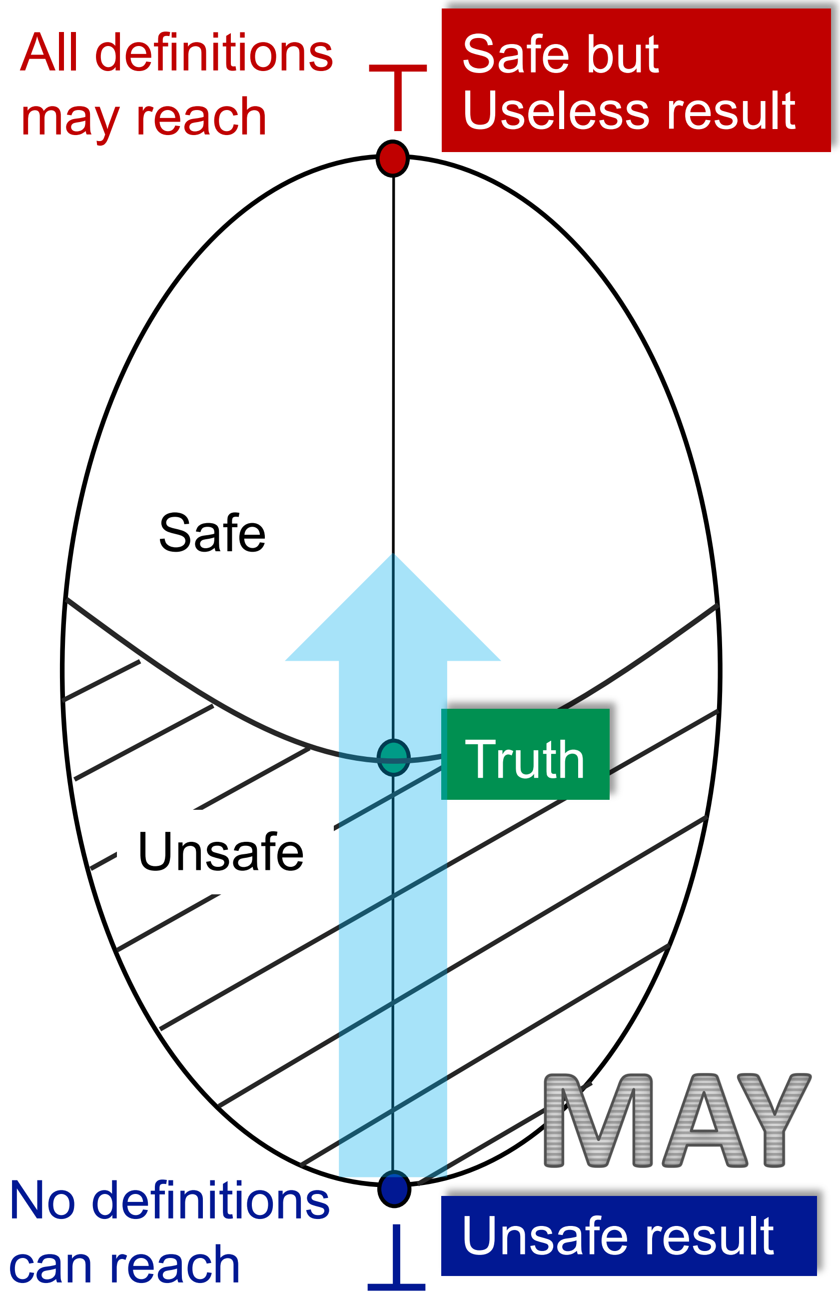
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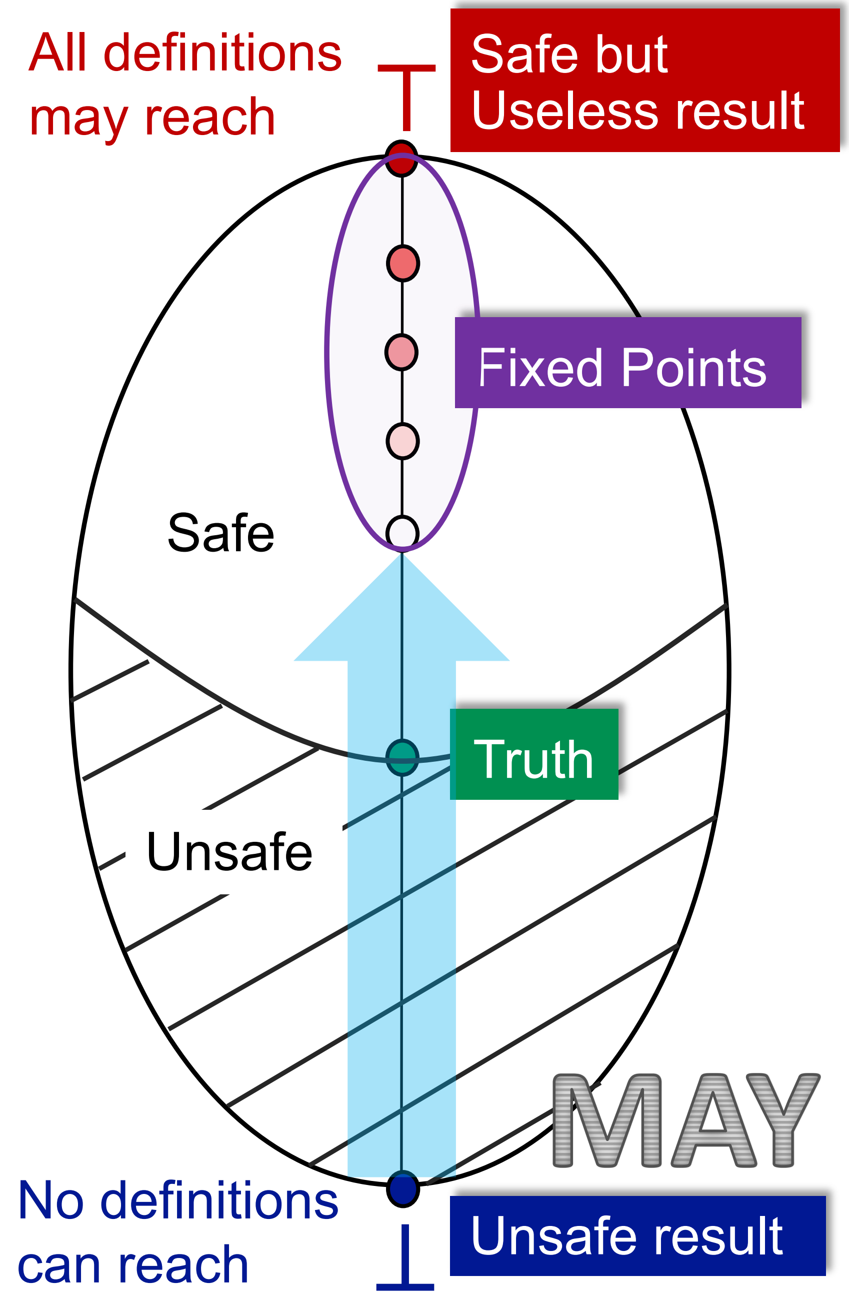
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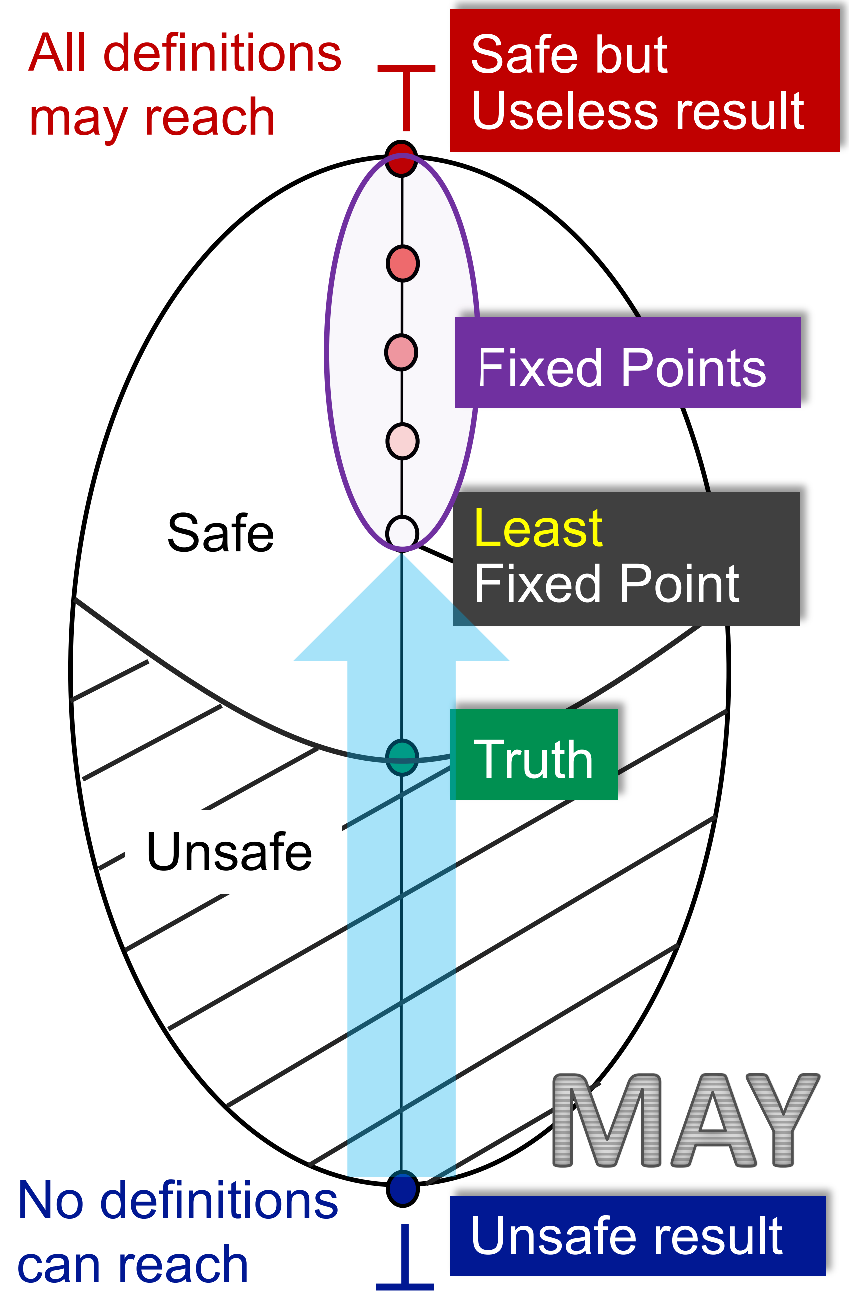
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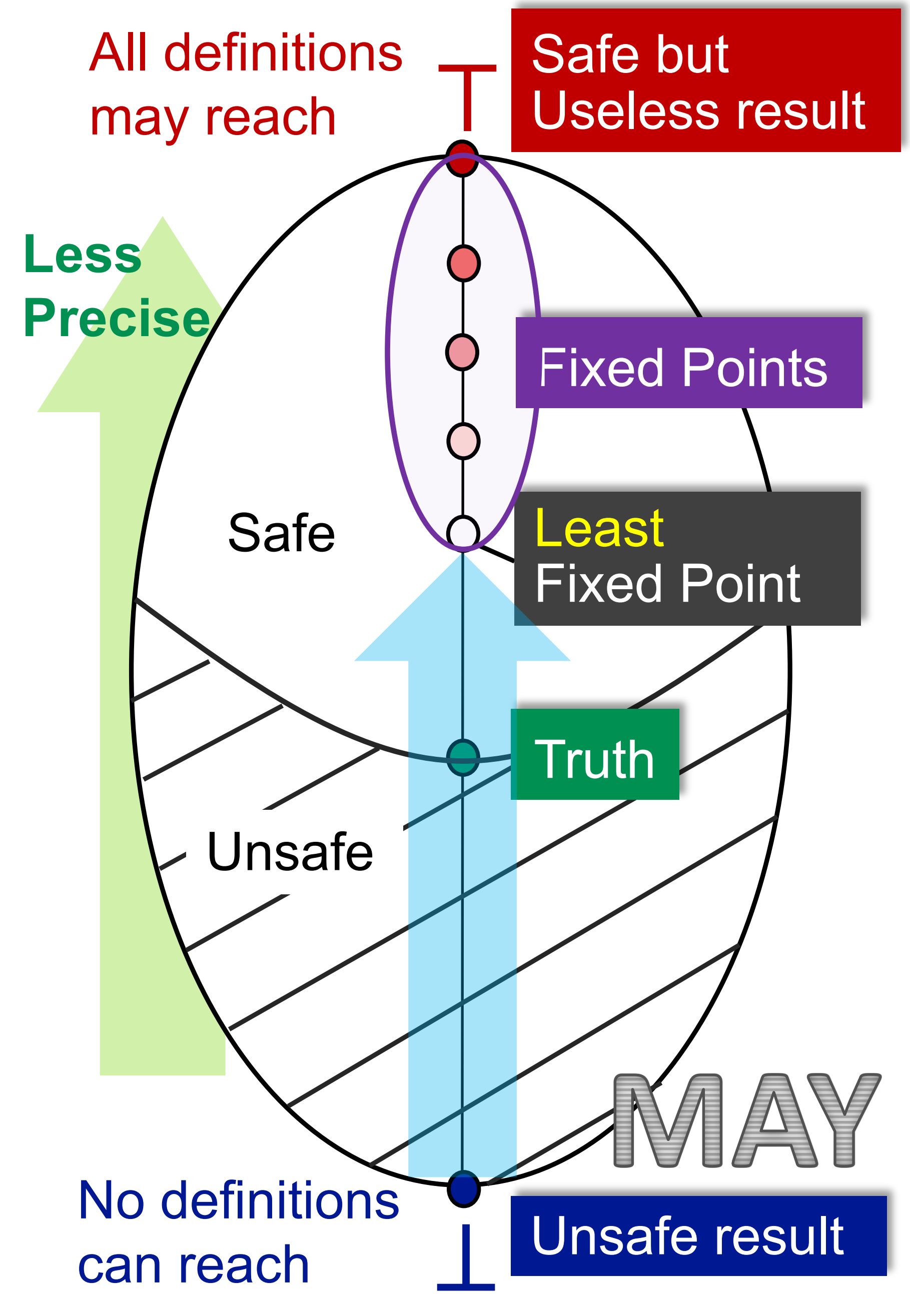

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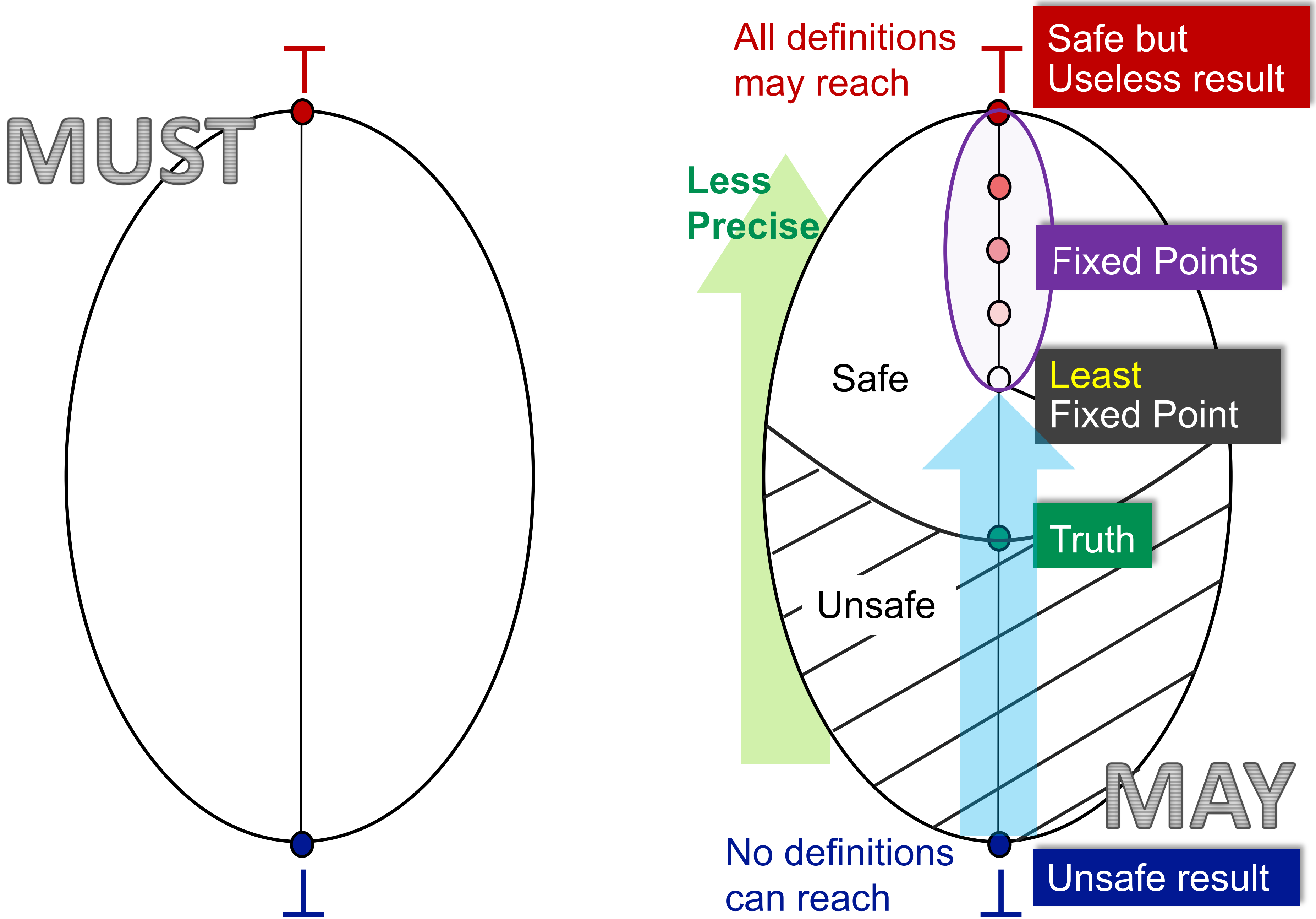
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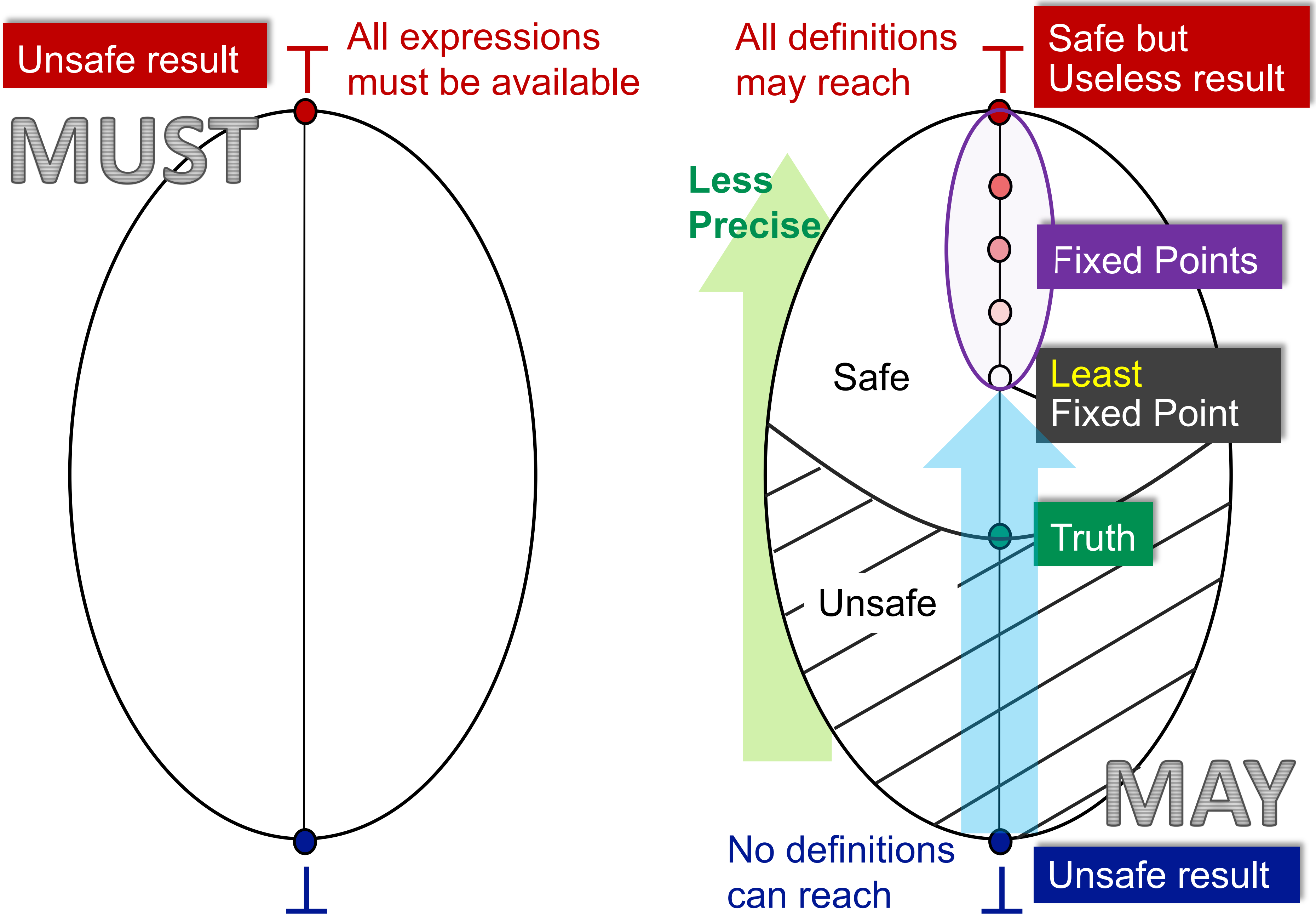
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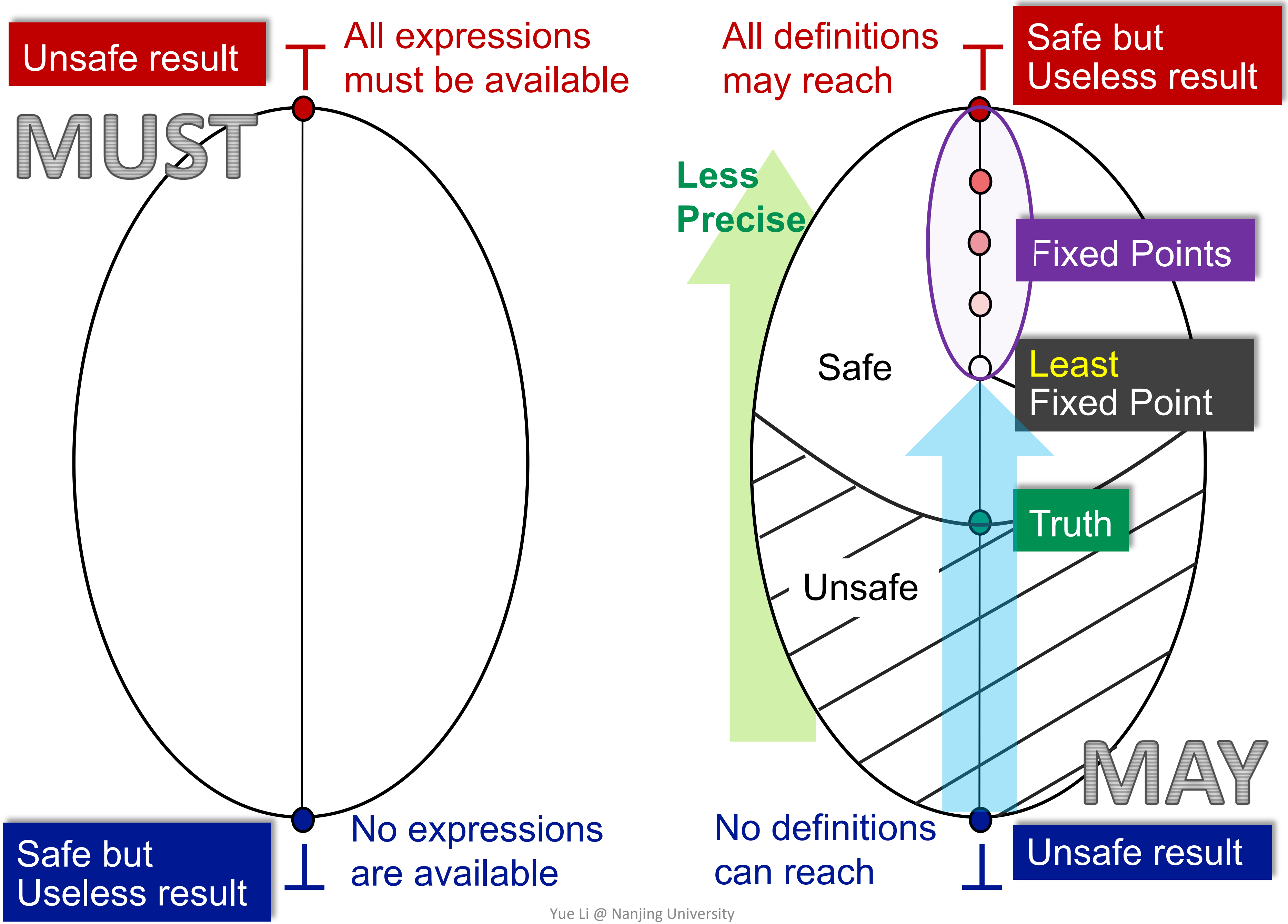

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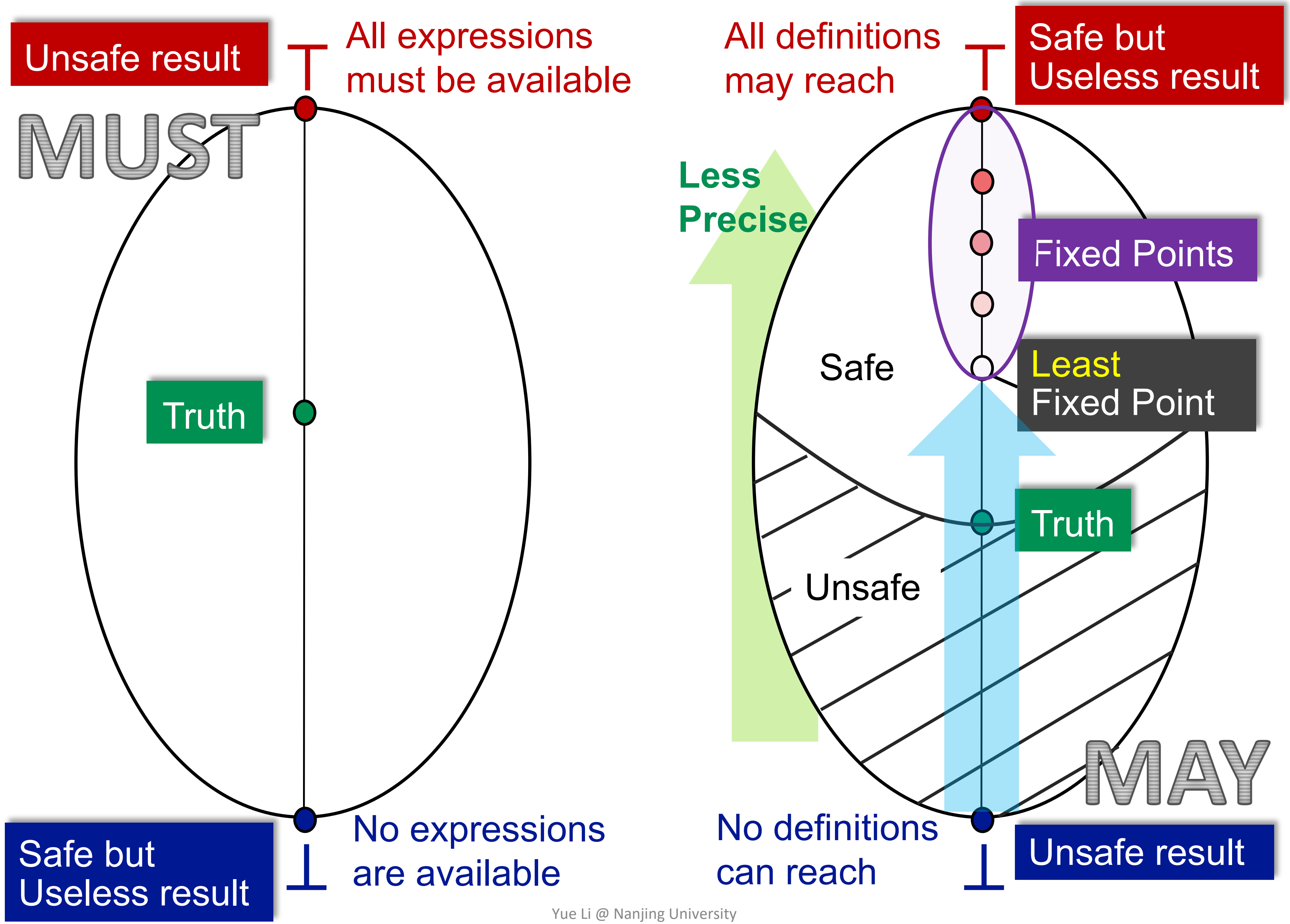
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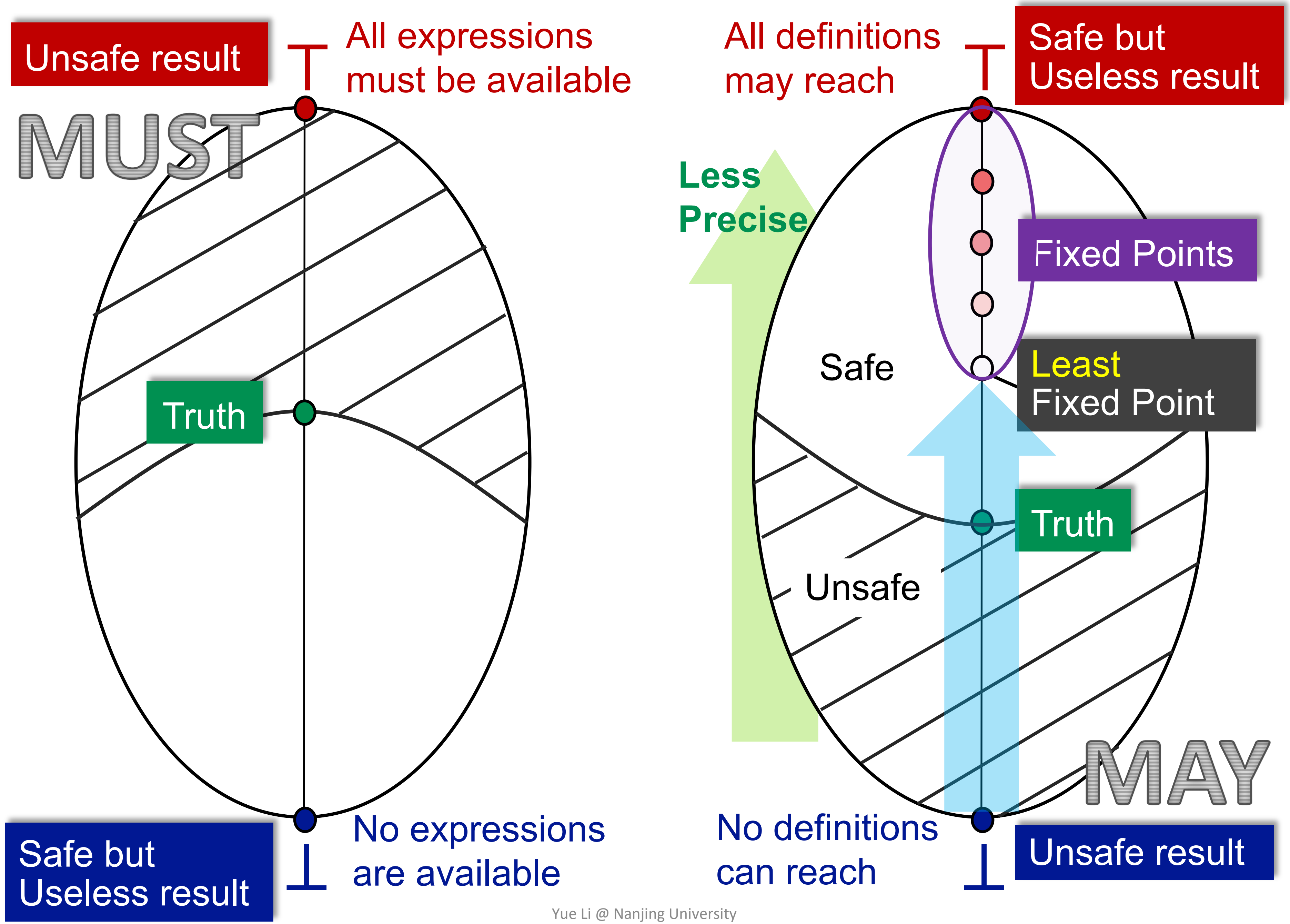
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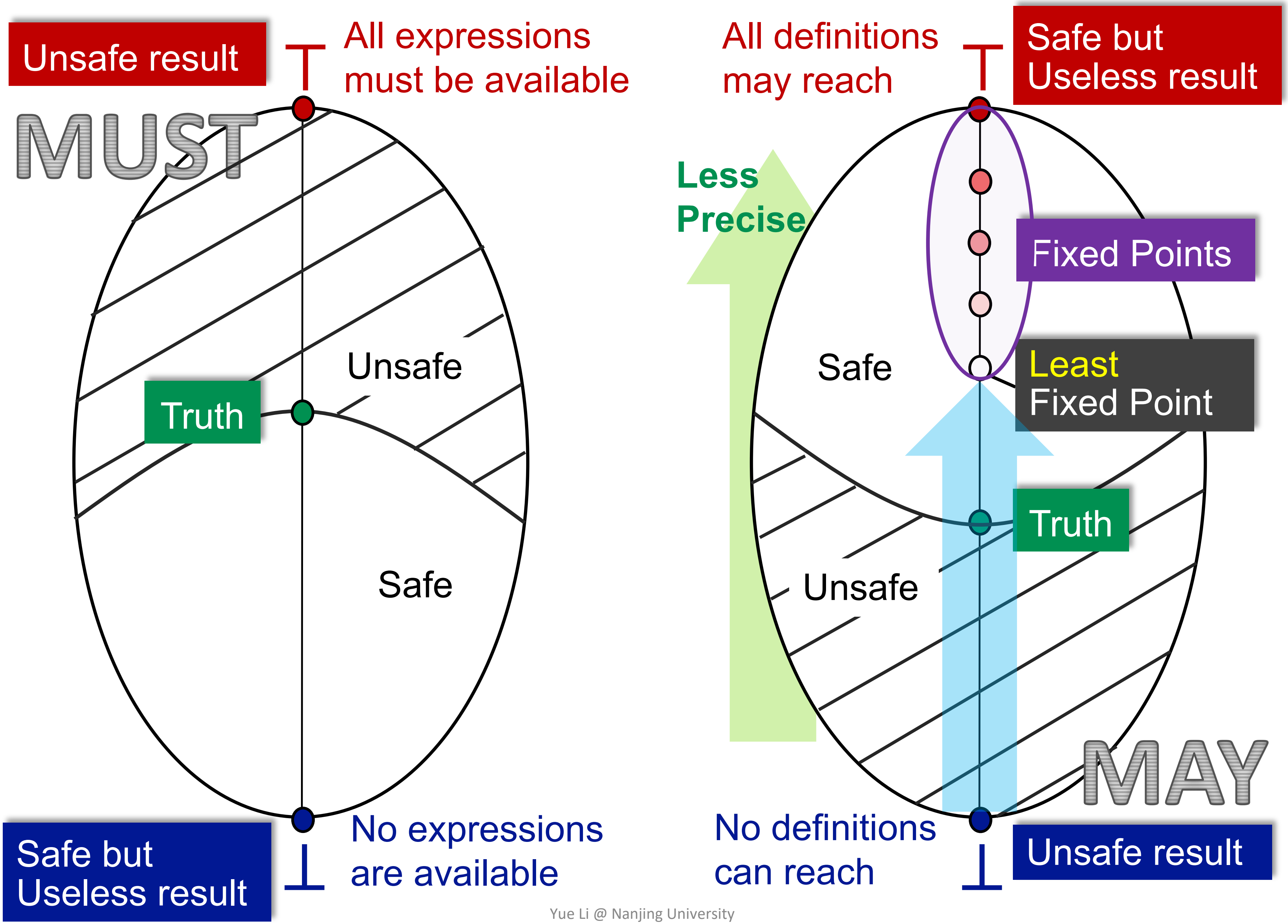

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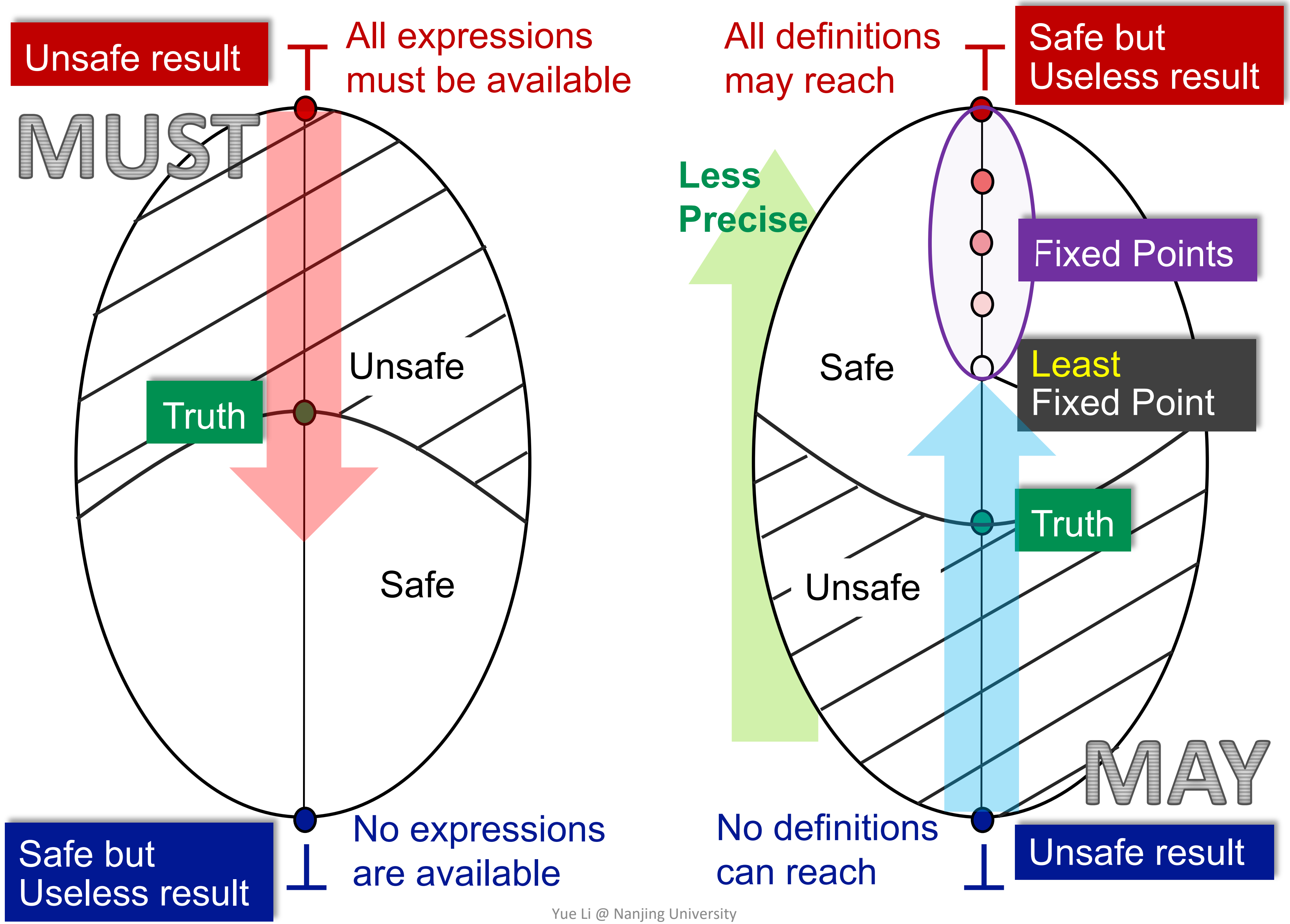

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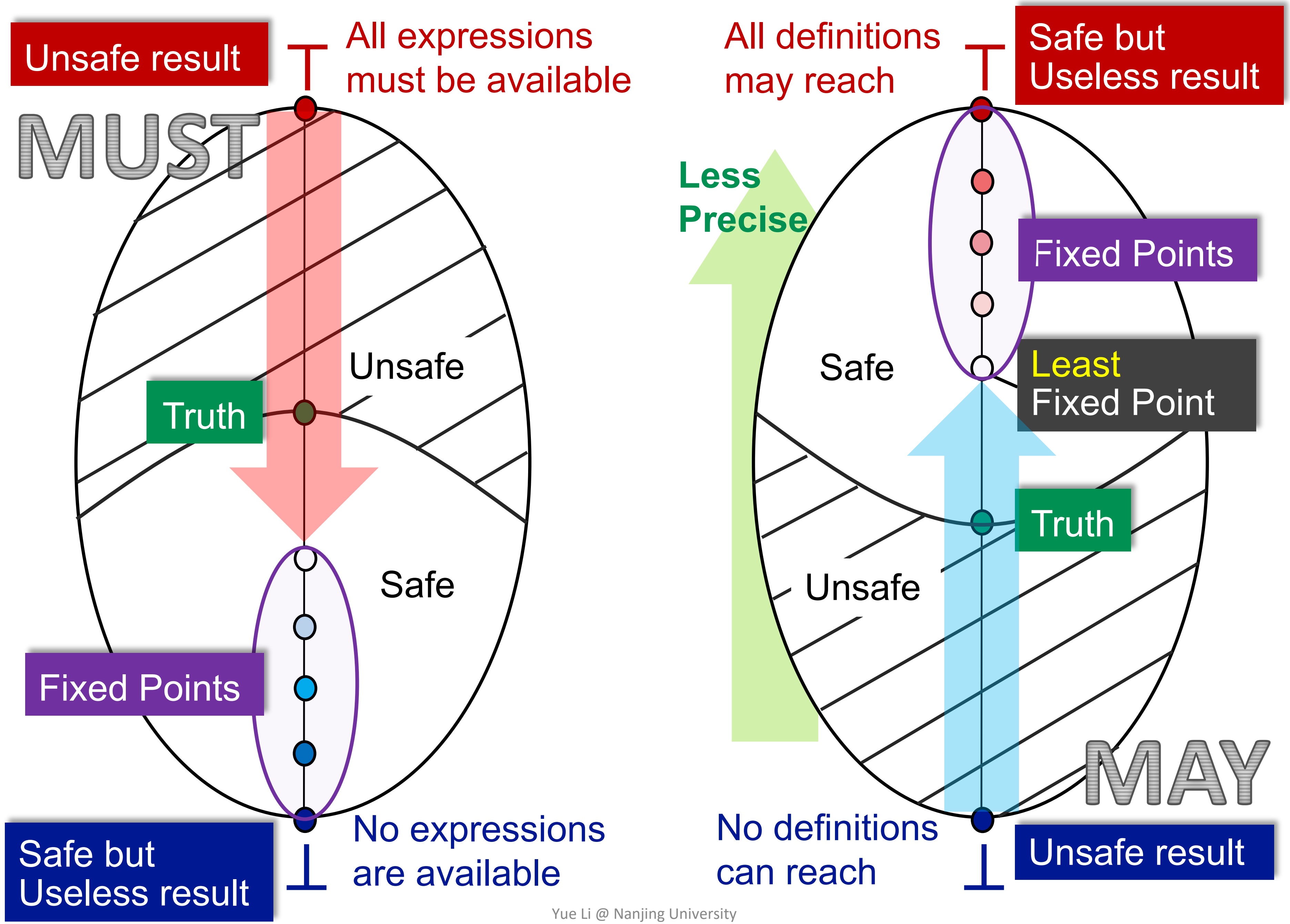

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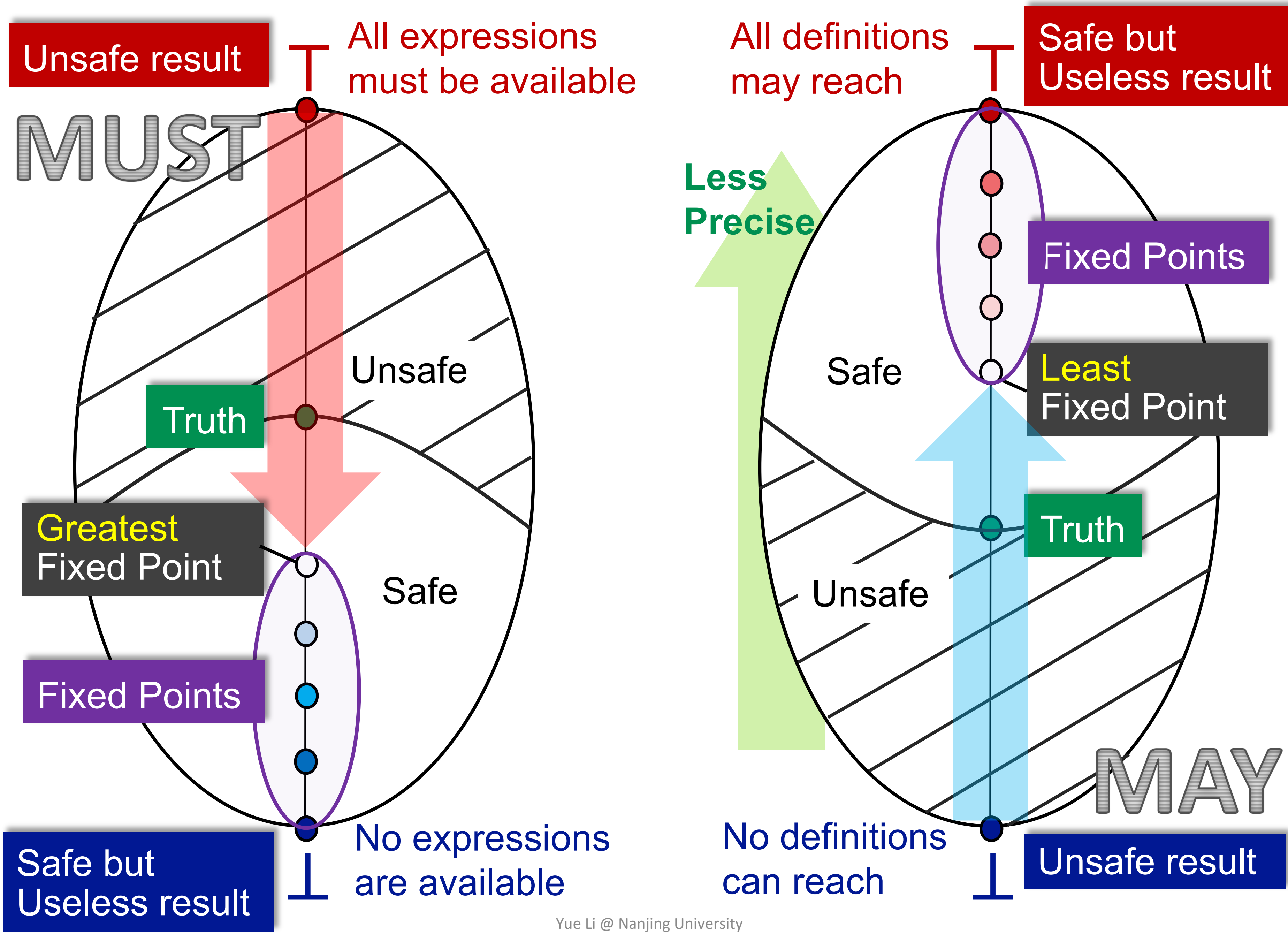
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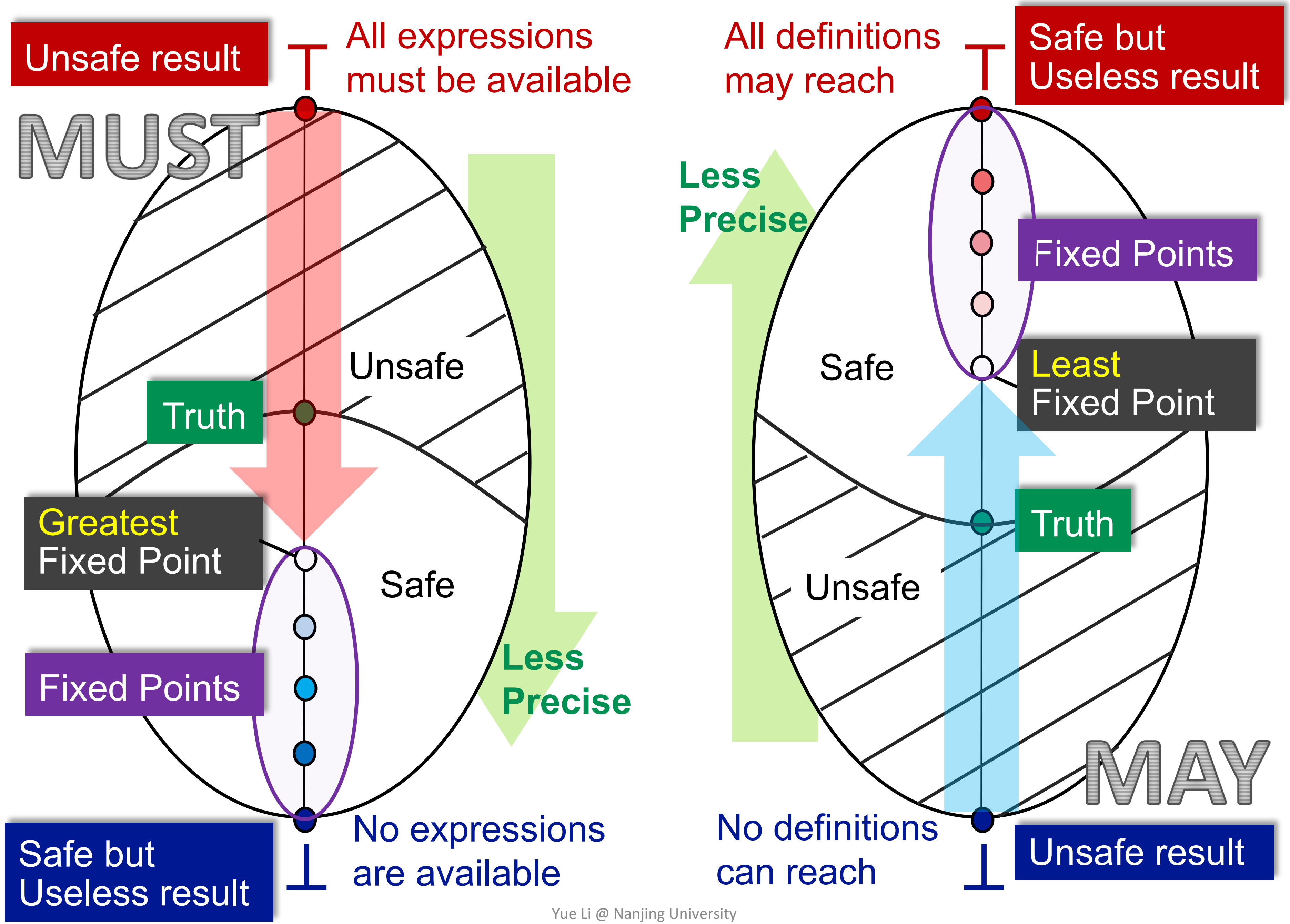

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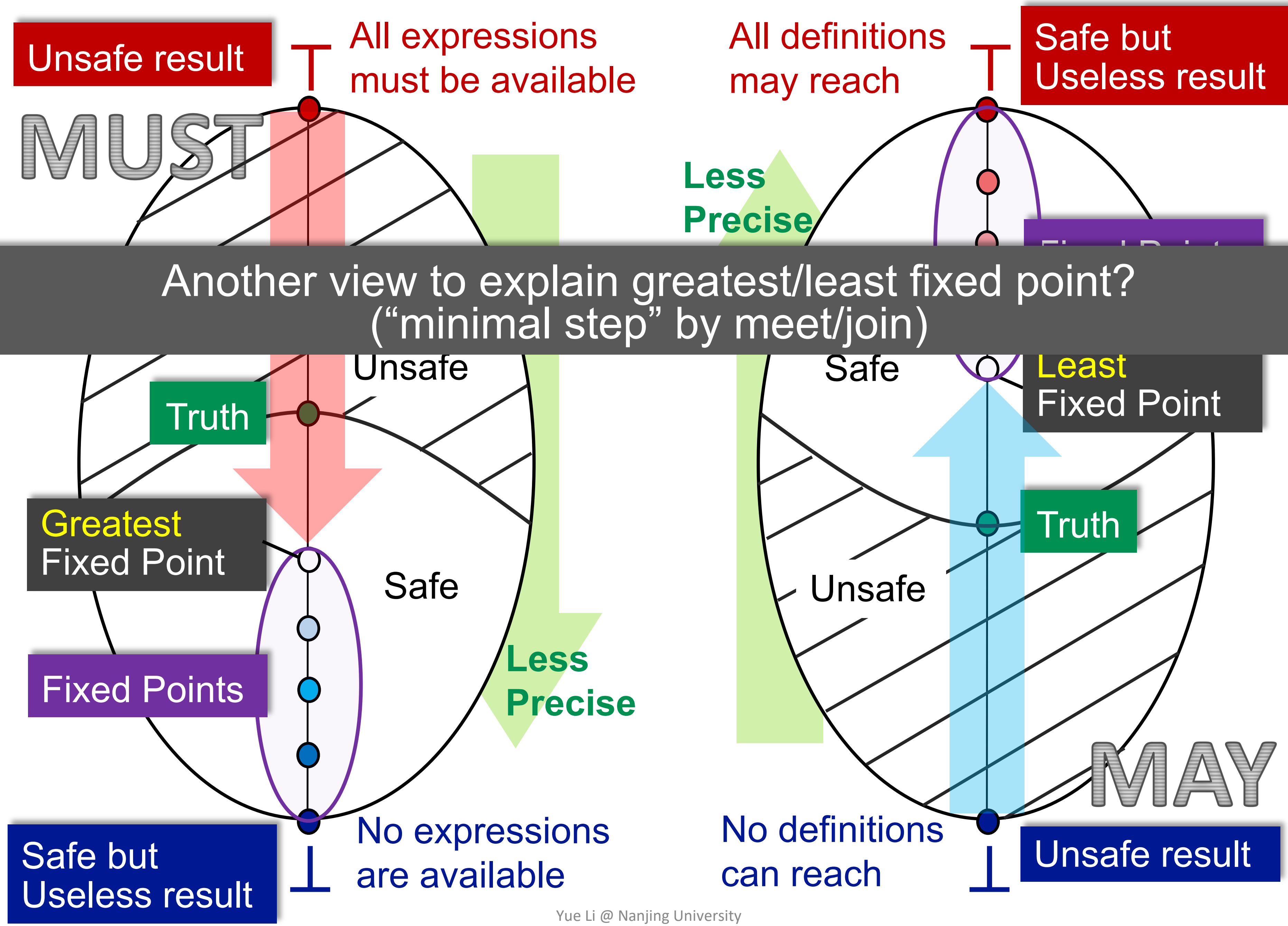
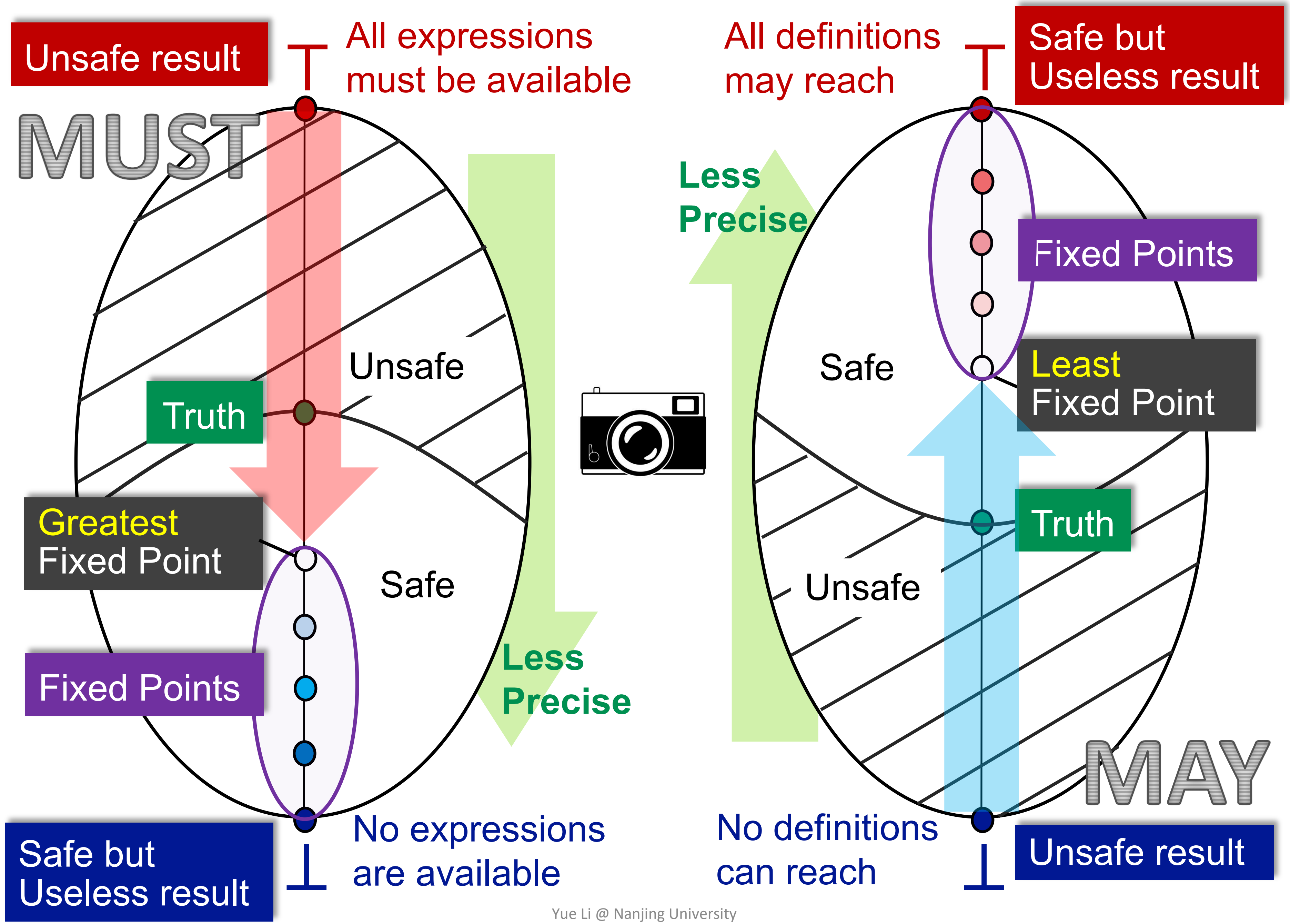


Another view to explain greatest/least fixed point?
("minimal step" by meet/join)



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How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

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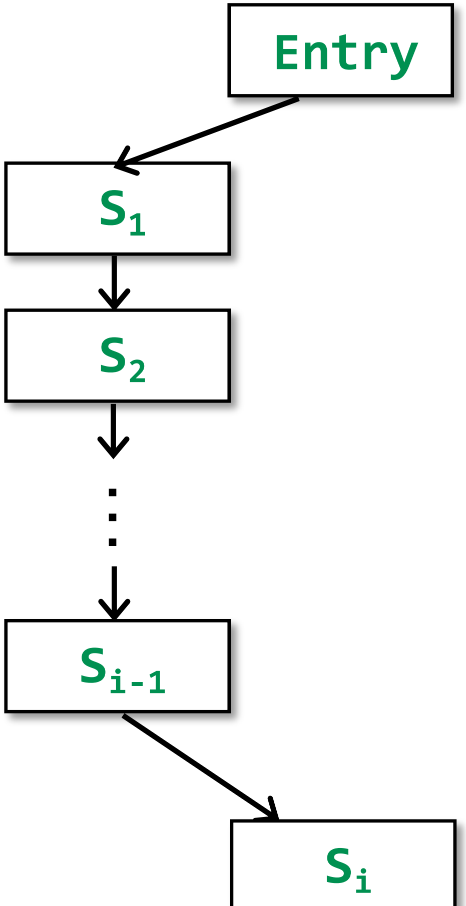

P = Entry → S_1 → S_2 → … → S_i

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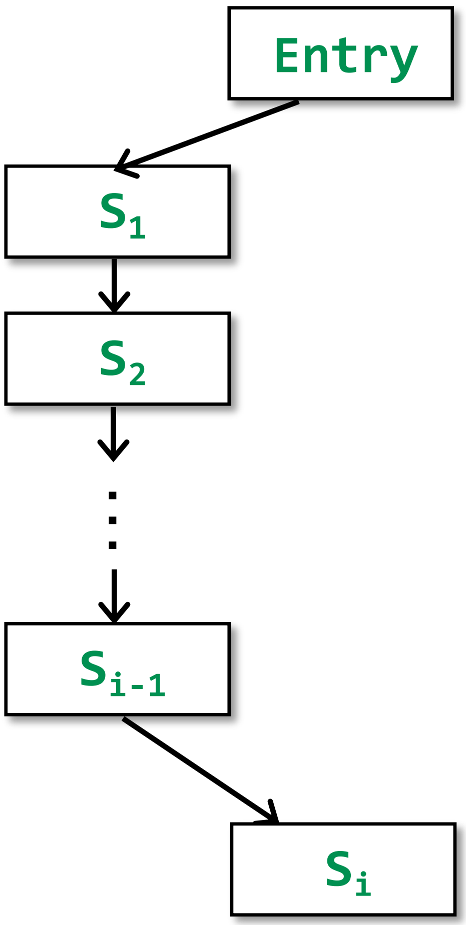
- Meet-Over-All-Paths Solution (MOP)



P = Entry → S_1 → S_2 → … → S_i

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

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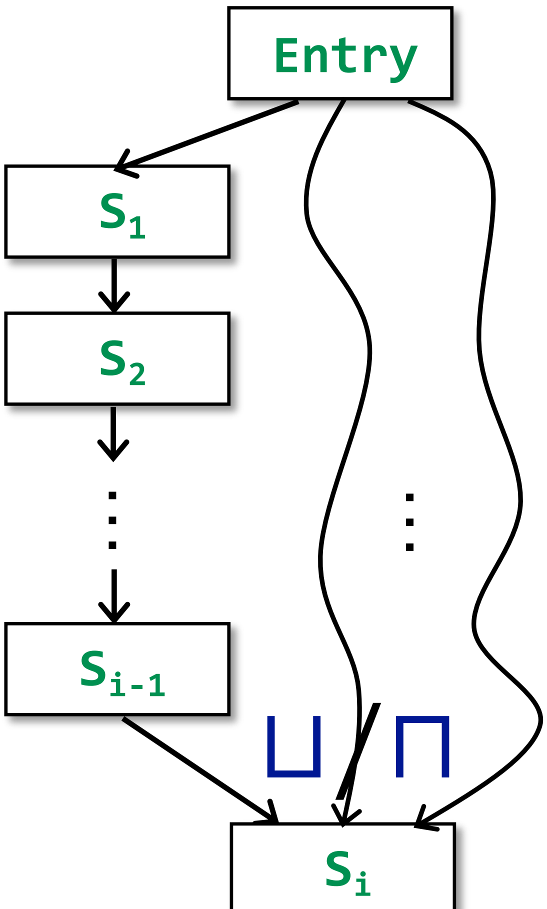

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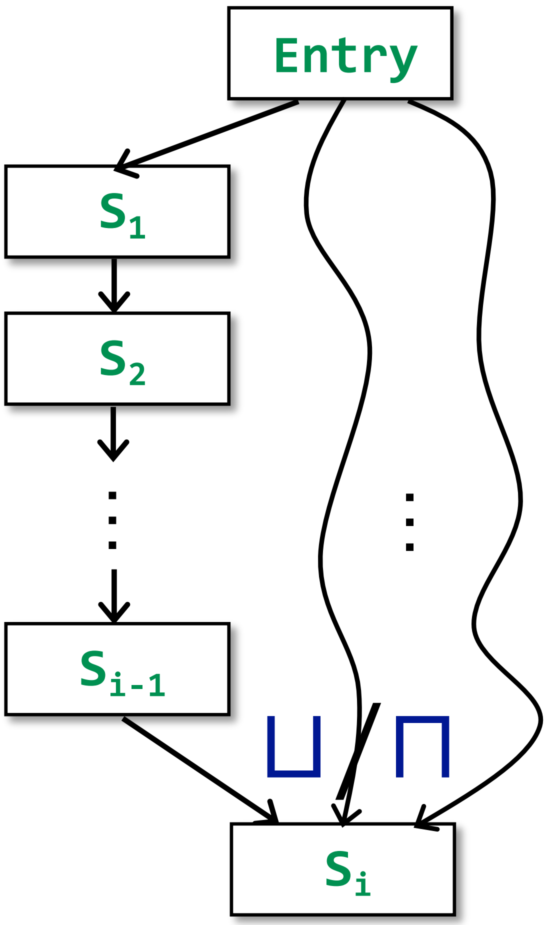

$P = Entry \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_i$

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: $f_{S1}, f_{S2}, \dots, f_{Si-1}$

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MOP computes the data-flow values at the end of each path and apply join / meet operator to these values to find their lub / glb

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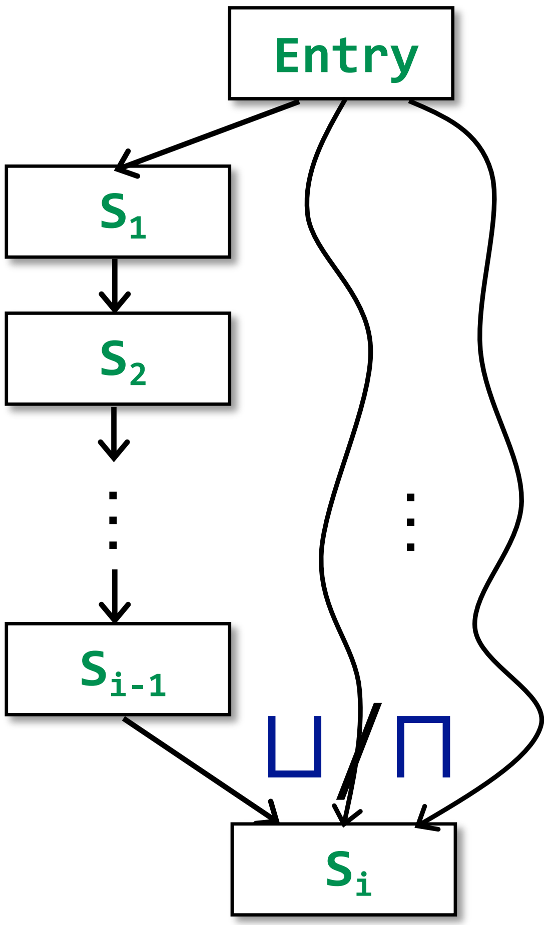

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Some paths may be not executable → **not fully precise**

How Precise Is Our Solution?

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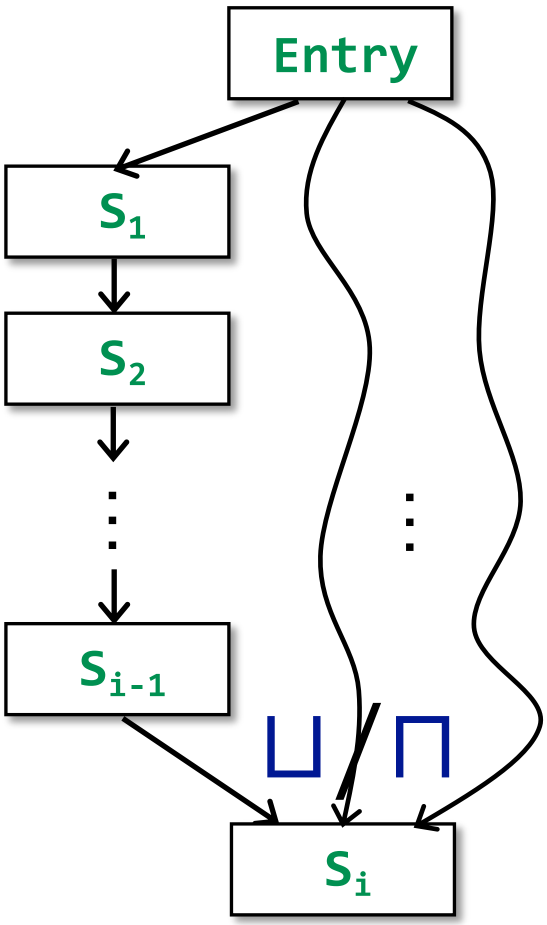

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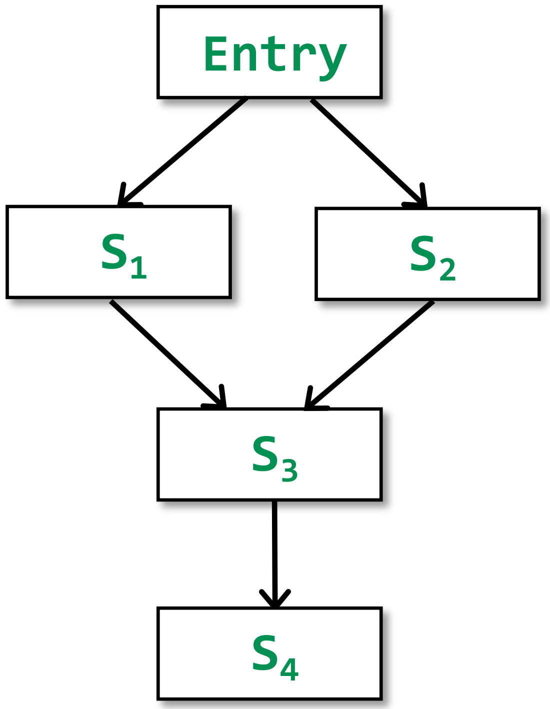
Some paths may be not executable → **not fully precise**

Unbounded, and not enumerable → **impractical**

Ours (Iterative Algorithm) vs. MOP



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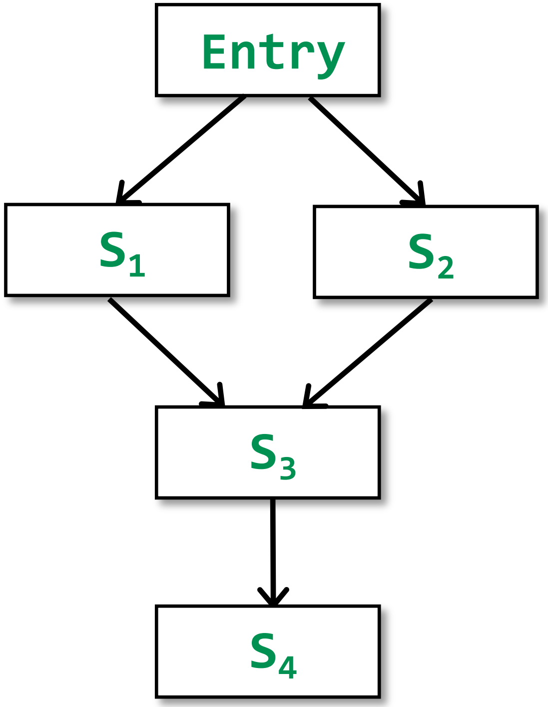

$$IN[S_4] = f_{s_3}(f_{s_1}(OUT[\texttt{Entry}]) \sqcup f_{s_2}(OUT[\texttt{Entry}]))$$

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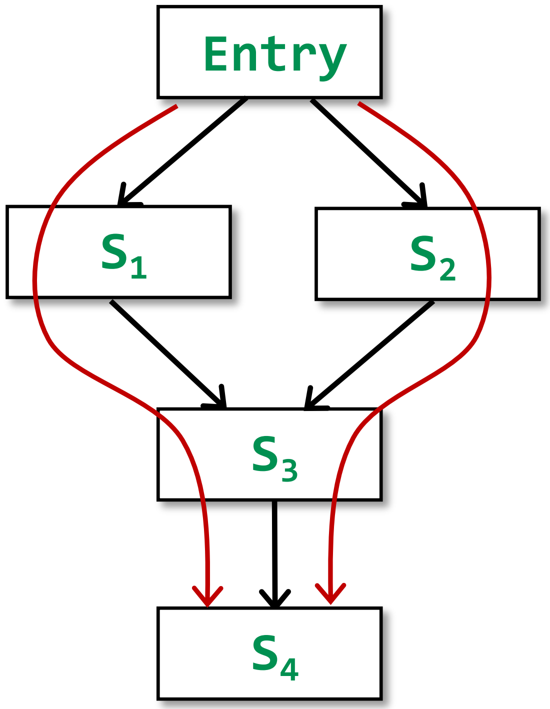

$$IN[s_4] = f_{s_3}(f_{s_1}(OUT[\texttt{Entry}]) \sqcup f_{s_2}(OUT[\texttt{Entry}]))$$

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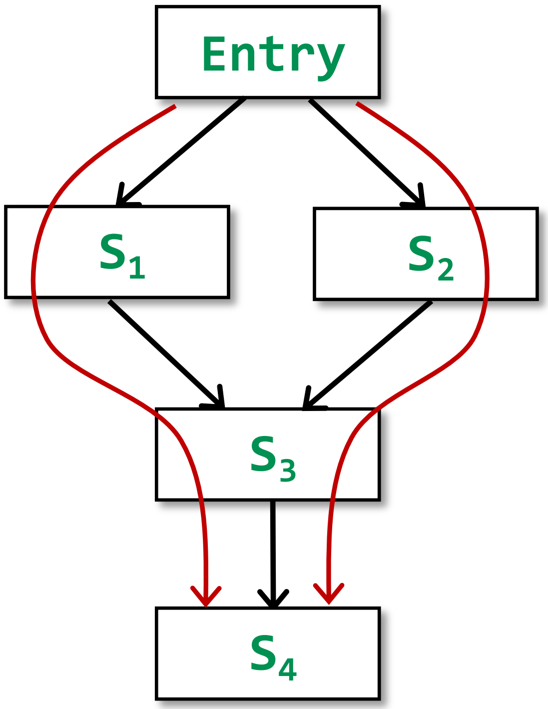

$$\mathrm{IN}[S_4] = f_{S_3}\left(f_{S_1}(\mathrm{OUT}[\mathtt{Entry}]) \sqcup f_{S_2}(\mathrm{OUT}[\mathtt{Entry}])\right)$$

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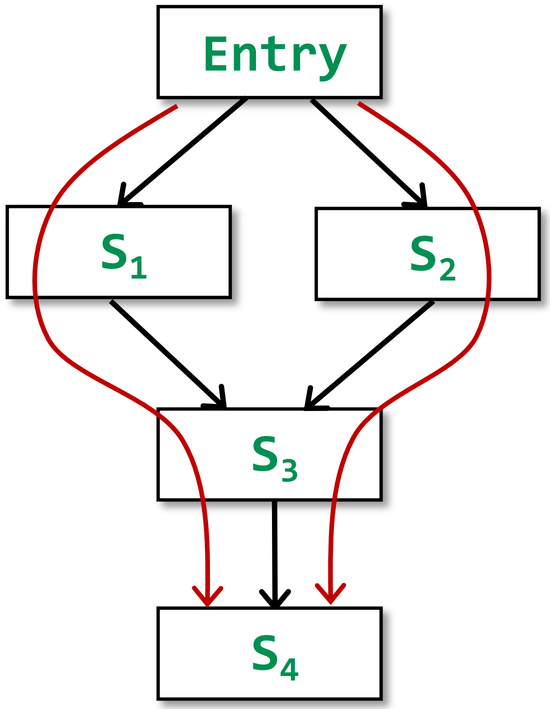

Ours = F(x ⊔ y)

MOP = F(x) ⊔ F(y)

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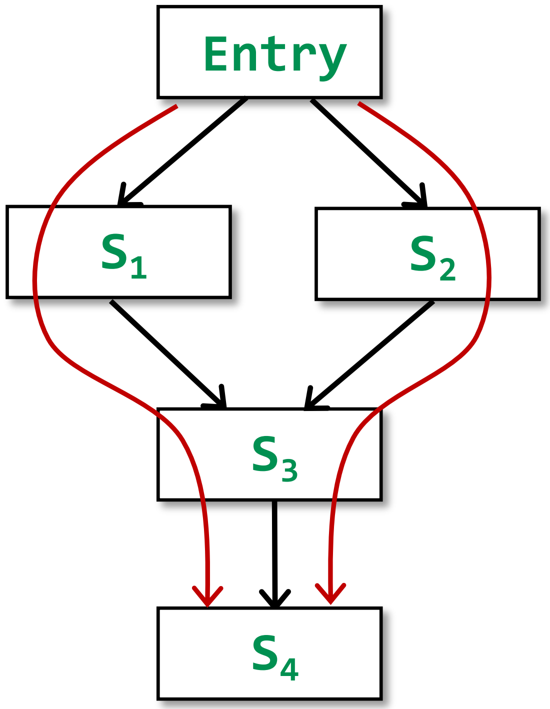

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?

$$\mathrm{IN}[s_4] = f_{s_3}(f_{s_1}(\mathrm{OUT}[\mathtt{Entry}]) \sqcup f_{s_2}(\mathrm{OUT}[\mathtt{Entry}]))$$

$$\mathrm{MOP}[s_4] = f_{s_3}(f_{s_1}(\mathrm{OUT}[\mathtt{Entry}])) \sqcup f_{s_3}(f_{s_2}(\mathrm{OUT}[\mathtt{Entry}]))$$

Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

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Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

MOP = $F(x) \sqcup F(y)$

By definition of lub $\sqcup$, we have

$$x \sqsubseteq x \sqcup y \text{ and } y \sqsubseteq x \sqcup y$$

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As transfer function F is **monotonic**, we have

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(Ours is less precise tha

Bit-vector or Gen/Kill problems (set union /intersection for join/meet) are distributive

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But some analyses are not distributive

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Given a variable x at program point p, determine whether x is guaranteed to hold a constant value at p.

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A data flow analysis framework (D, L, F) consists of:

- **D**: a direction of data flow: forwards or backwards
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Constant Propagation – Lattice

- Domain of the values V

- Meet Operator ⊓

Constant Propagation – Lattice

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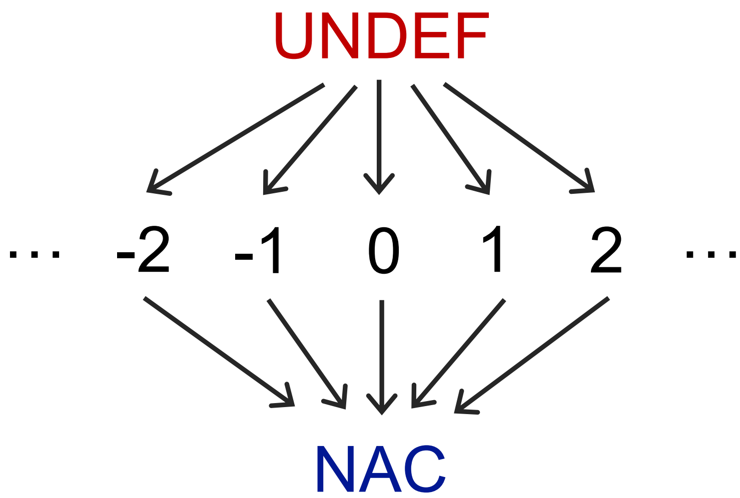

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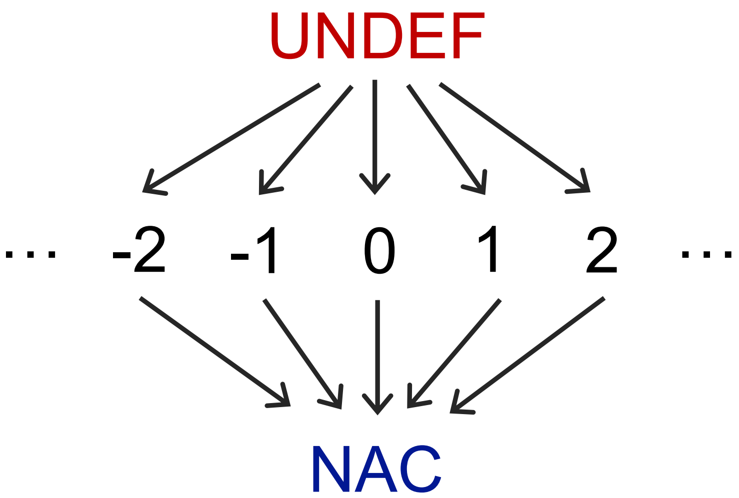

- Meet Operator ⊓

NAC ⊓ v = NAC

Constant Propagation – Lattice

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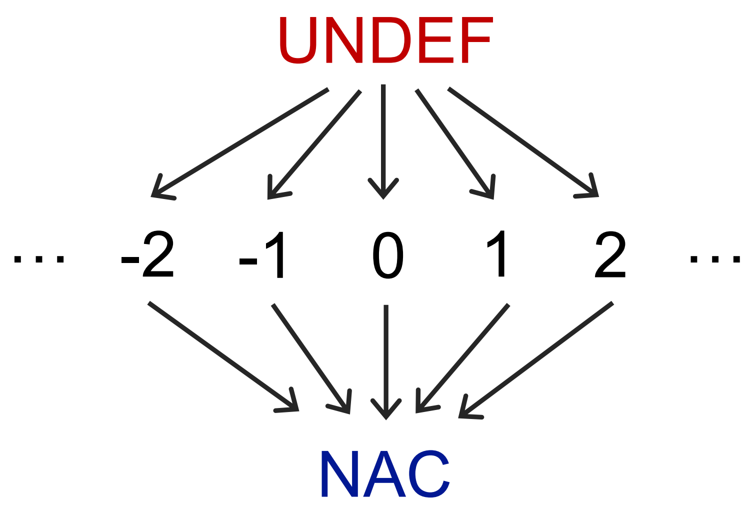

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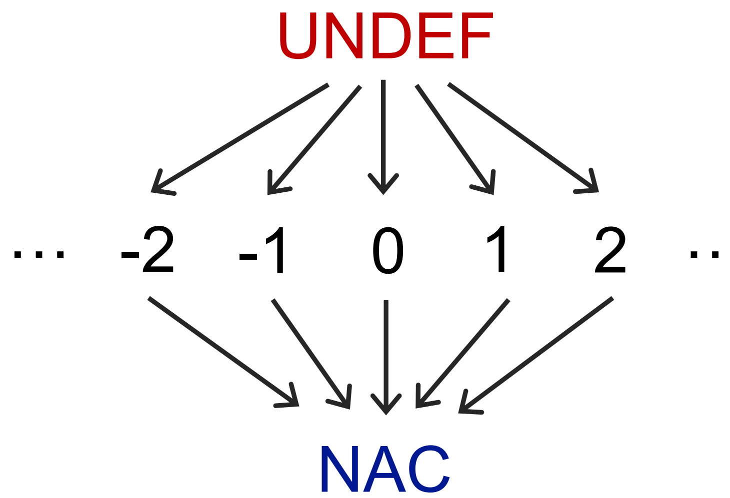

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Uninitialized variables are not the focus in our constant propagation analysis

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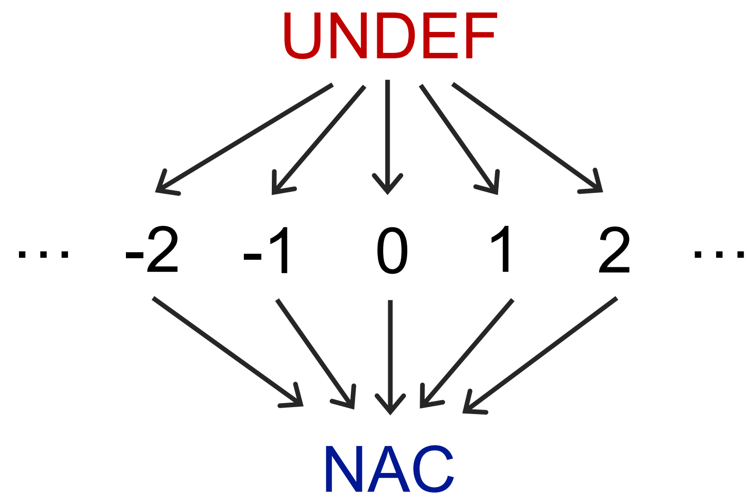

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c ⊓ v = ?

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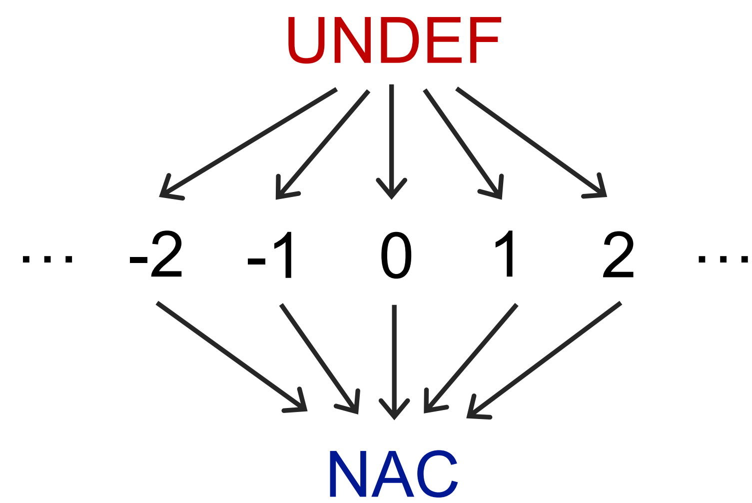

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- $c \sqcap c = c$
- $c_1 \sqcap c_2$ = NAC

Constant Propagation – Lattice

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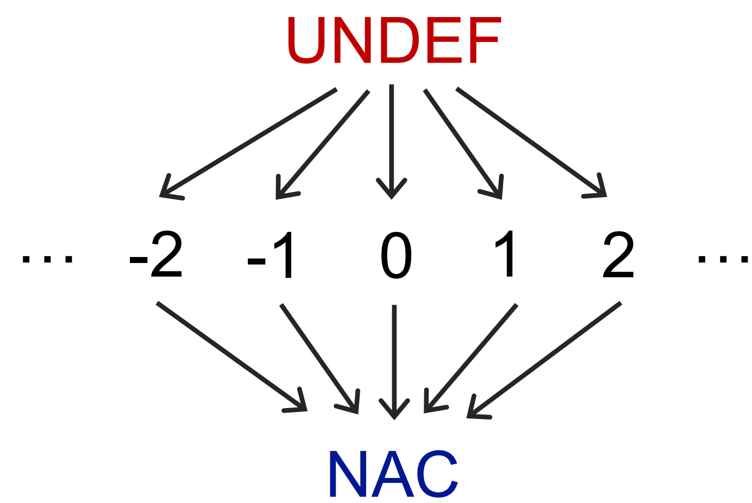

- Meet Operator $\sqcap$

NAC $\sqcap$ v = NAC

UNDEF $\sqcap$ v = v

Uninitialized variables are not the focus in our constant propagation analysis

$c \sqcap v$ = ?

- $c \sqcap c = c$
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At each path confluence PC, we should apply “meet” for all variables in the incoming data-flow values at that PC

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Given a statement s: x = …, we define its transfer function F as

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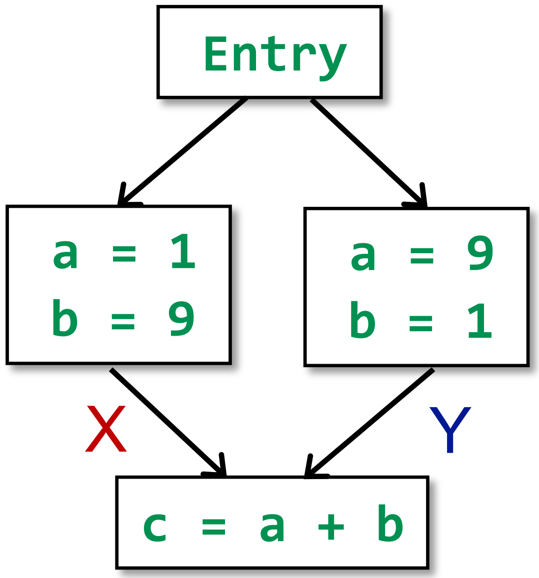
(if s is not an assignment statement, F is the identity function)

Constant Propagation – Nondistributivity



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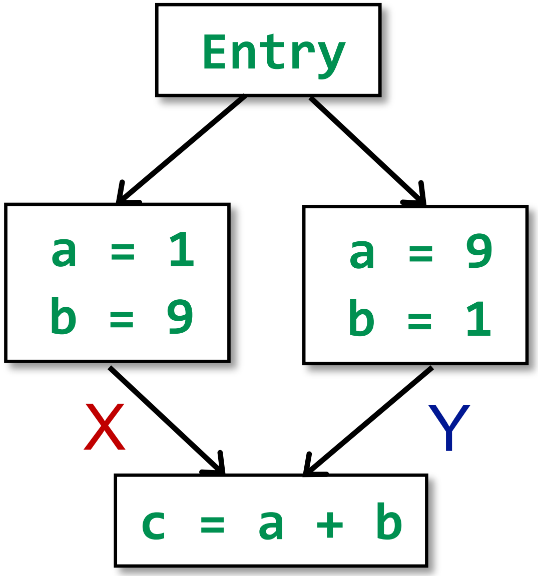

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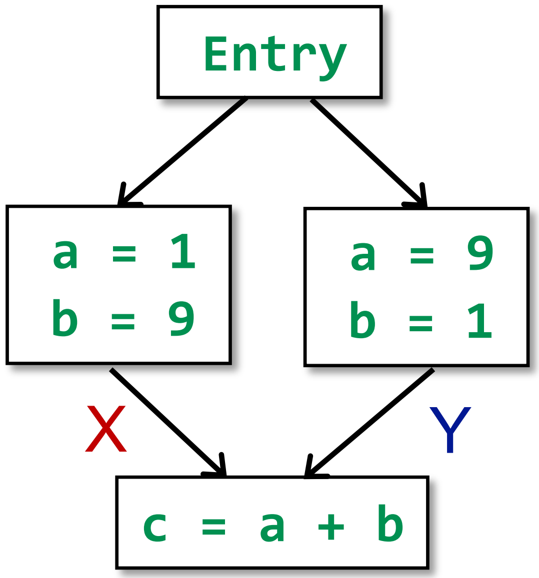

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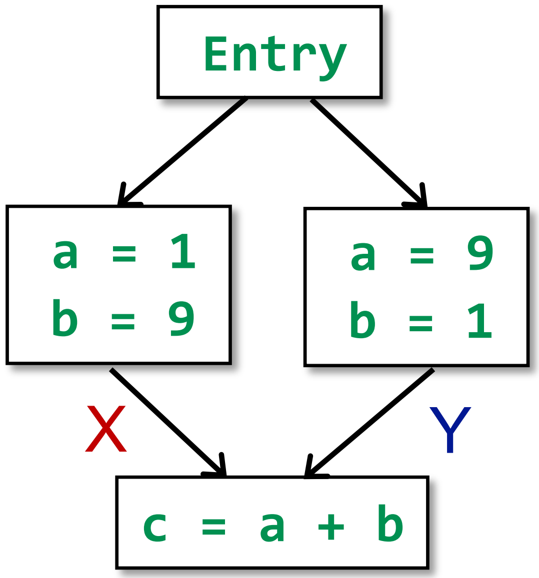

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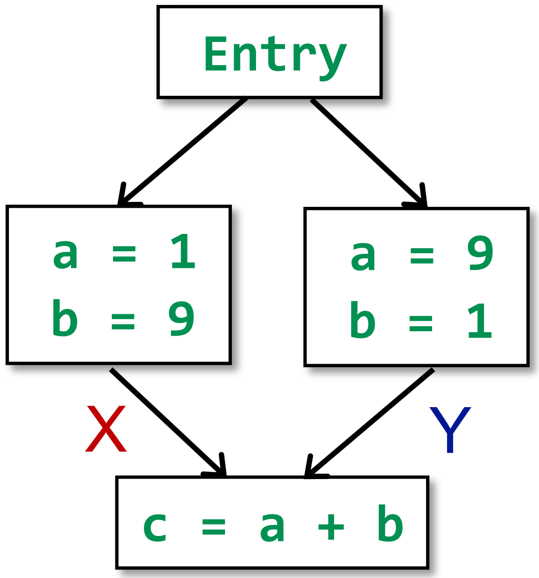

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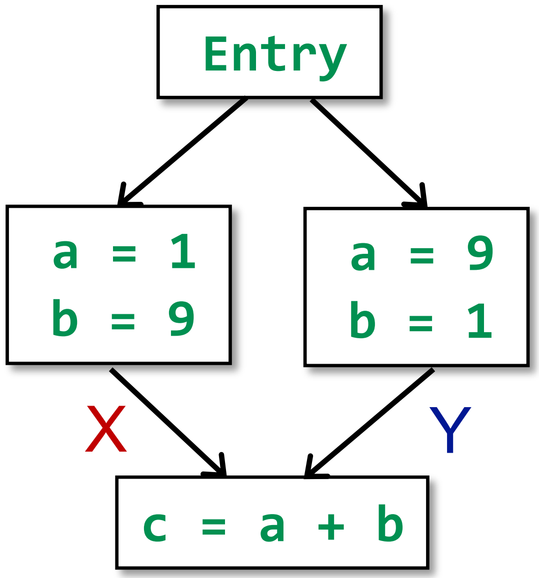

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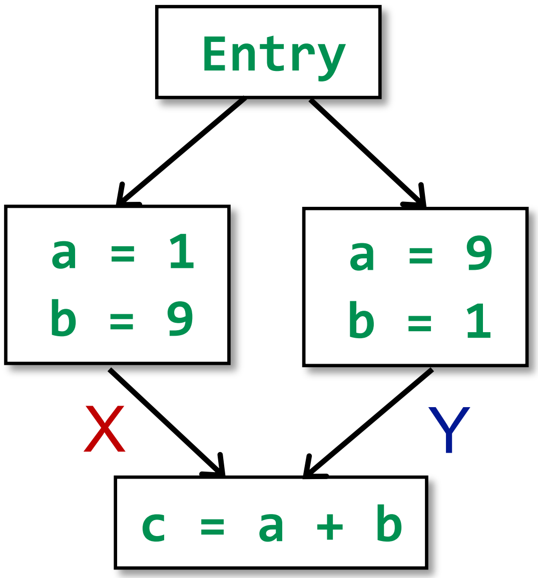

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Show our constant propagation analysis is monotonic

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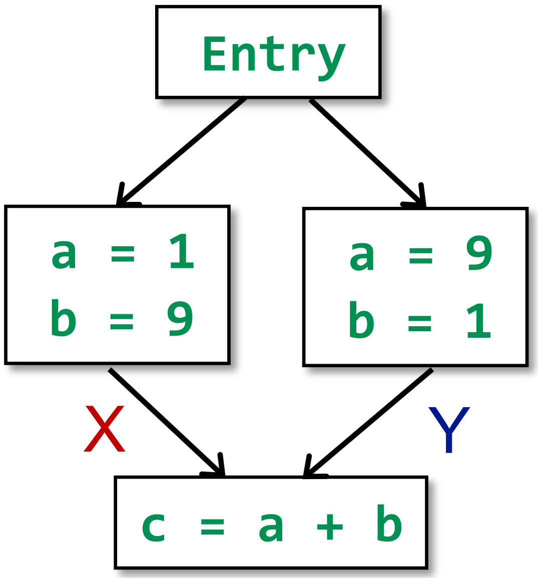

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Worklist Algorithm,

an optimization of Iterative Algorithm

Review Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block B)

OUTPUT: IN[B] and OUT[B] for each basic block B

METHOD:

OUT[*entry*] = ∅;
for (each basic block $B \backslash entry$)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block $B \backslash entry$) {
        IN[B] = $\bigsqcup_{P \text{ a predecessor of } B}$ OUT[P];
        OUT[B] = $gen_B \cup (\text{IN}[B] - kill_B)$;
    }

Worklist Algorithm

Forward Analysis

OUT[*entry*] = ∅;
for (each basic block *B**entry*)
    OUT[*B*] = ∅;
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block *B* from Worklist
    old_OUT = OUT[*B*]
    IN[*B*] = $\bigsqcup_{P \text{ a predecessor of } B}$ OUT[*P*];
    OUT[*B*] = gen_B U (IN[*B*] - $kill_B$);
    if (old_OUT ≠ OUT[*B*])
        Add all successors of *B* to Worklist

Worklist Algorithm

Forward Analysis

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    Pick a basic block *B* from Worklist
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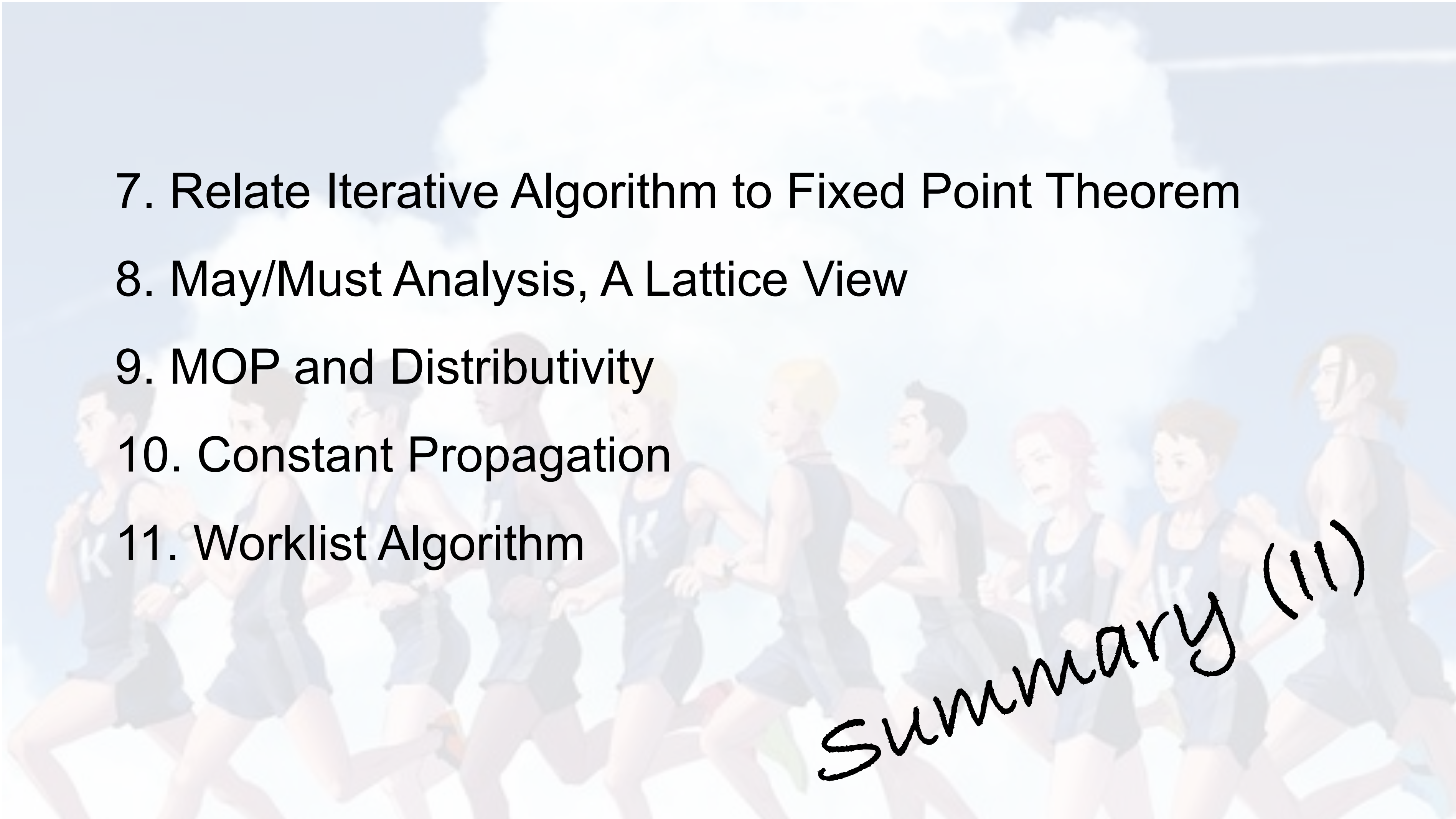
    $\mathrm{IN}[B] = \bigsqcup_{P \text{ a predecessor of } B} \mathrm{OUT}[P];$
    $\mathrm{OUT}[B] = gen_B \cup (\mathrm{IN}[B] - kill_B);$
    if (old_OUT ≠ OUT[*B*])
        Add all successors of *B* to Worklist

OUT will not change if IN does not change

Summary (I)

1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem



Summary (II)

7. Relate Iterative Algorithm to Fixed Point Theorem
8. May/Must Analysis, A Lattice View
9. MOP and Distributivity
10. Constant Propagation
11. Worklist Algorithm

The X You Need To Understand in This Lecture

- Understand the functional view of iterative algorithm
- The definitions of lattice and complete lattice
- Understand the fixed-point theorem
- How to summarize may and must analyses in lattices
- The relation between MOP and the solution produced by the iterative algorithm
- Constant propagation analysis
- Worklist algorithm

Assignment Two:
Constant propagation and worklist solver